Painting Perfect Polynomials

Calculus 11/12, Veritas Prep.

Polynomials, as you have known for years, are functions that look like

$$f(x) = 5x^2 + 3x - 2 \qquad f(x) = -8x^{15} \qquad f(x) = 3x + 4$$
$$f(x) = 25.7x^{9{,}000{,}000} - 3x \qquad f(x) = 9 \qquad f(x) = x^3 + 92x$$

and so on. Basically, we have a bunch of terms added together, where each term looks like a variable raised to some power times some coefficient. We can have as many or as few terms as we want; the coefficients can be integers, real numbers, whatever.

The only real restriction is that the variable must be raised to some non-negative, integer power. For example, as tempting as it is to say that something like $5x^{-2} - 3x^{-1}$ is a polynomial, it's not. Sorry. We exclude things with negative exponents from our definition of a polynomial because they cause problems—we get vertical asymptotes and such, which are messy and discontinuous, and we want to be able to live in a world without discontinuities. Such is the world of polynomials.

Likewise, we don't want to call something with fractional exponents, like $5x^{1/2} - 3x^{2/3}$, a polynomial, either. Imagine $x^{1/2}$—a square root. What if you try to plug a negative in (i.e., take the square root of a negative number)? You can't do it. (Not in the real numbers, anyway.) We don't want that to be a polynomial—we want to be able to plug any number, positive or negative, big or small, into our polynomial, and get a result.

What about something like $x^{1/3}$? This is just another way of writing a cube root. And we can take the cube root of a negative number—the domain of $x^{1/3}$ is all real numbers. Except... what's going on at $x = 0$? The function has a vertical tangent line. (Think about what the graph looks like.) Put differently, the function, at the instant that $x = 0$, is vertical. (But not so much that it violates the vertical line test... crazy, eh?) And the immediate consequence of that is that its derivative (which is, after all, just the slope of the function) has a vertical asymptote there. Ouch. We don't want that—we want the function to have no vertical asymptotes, and we want its derivatives to have no vertical asymptotes, either. Not only do we want functions that behave nicely on the surface, we want them to still behave nicely once we start doing calculus with them (we want to be able to take their derivatives, as many derivatives we want, and still not get anything messy).

What else? Trigonometric and exponential functions (e^x, $\sin(x)$, etc.) aren't polynomials either. What we'll see eventually in calculus is that if we could write polynomials with infinitely many terms, we could write trig functions and exponentials as polynomials[1]. But that's for later. For now, we'll require our polynomials to be of finite length because infinity is nasty and nebulous and it's not entirely clear what we mean by "infinity"[2].

My point with all of this is that the definition of a polynomial is not something chiseled in stone that ancient mathematicians found and which has been passed down, generation to generation, ever since. The reason we define polynomials the way we do is because we want to have a class of functions that, to put it colloquially, *behave nicely.* We want to have functions that are defined everywhere—functions free of vertical asymptotes, holes, or other nether regions. We want to have functions that are differentiable everywhere—and whose derivatives are also differentiable everywhere, and also have no unpleasant features. This is why we want this thing we call a "polynomial" to satisfy all these conditions about the exponents being positive integers and so forth—because we want to create a mathematical playground we know we can have fun in without getting hurt.

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[1] And that in fact, trig functions *are just a kind of exponential function.*

[2] That is not to say that infinity is totally un-understandable—just that we're not yet at the point where we can. For a great book on this subject (my favorite popular math book, actually) see David Foster Wallace, *Everything and More: A Compact History of* ∞ (W.W. Norton, 2003).

Formally, we define a **polynomial** (written in **standard form**) as a function of the form:

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + c_{n-2} x^{n-2} + \cdots + c_2 x^2 + c_1 x^1 + c_0 x^0$$

or just

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + c_{n-2} x^{n-2} + \cdots + c_2 x^2 + c_1 x + c_0$$

where n is a nonnegative integer and where the set $\{c_n, c_{n-1}, c_{n-2}, \ldots, c_2, c_1, c_0\}$ are real numbers (called the **coefficients** of each term). The subscripts are just a way of labeling each of the coefficients—you could label them as a, b, c etc., if you wanted, but it'd make it hard to write the definition (if you start with $ax^n + bx^{n-1} \ldots$, what would you call the coefficient on the x^1 and x^0 terms?) Note that when we write them as $c_n, c_{n-1}, \cdots$, the n and $n-1$ have no connection to the actual, numerical value of the coefficient—they're just a way of saying, "this number c_k (for some k) is the number I want to be the coefficient of the x^k term." For example, maybe you have a third degree polynomial like $4x^3 + 8x^2 + 7x + 1.4$. Then $c_3 = 4$, $c_2 = 8$, $c_1 = 7$, and $c_0 = 1.4$.

If you like, you can write the definition as a finite sum[3]:

$$f(x) = \sum_{k=0}^{k=n} c_k x^k$$

There's some terminology associated with polynomials. The value of the highest exponent of the polynomial—in this case, n—is called the **degree** of the polynomial. The term with the highest exponent is called the **leading term** (because it's usually written first), and the coefficient on that term is called the **leading coefficient**, like so:

$$f(x) = \overbrace{c_n x^n}^{\text{leading term}} + \underbrace{c_{n-1} x^{n-1}}_{\text{term}} + \underbrace{c_{n-2} x^{n-2}}_{\text{another term}} + \cdots + c_2 x^2 + c_1 x + \underbrace{c_0}_{\text{constant term}}$$

Often the terms between the leading term and the constant term are called the **cross terms**.

We can also write a polynomial in **factored form**, which is often more convenient. (Sometimes this is also called the **complete factorization** of the polynomial.) In fact, for sketching polynomials, it *is* more convenient, since it makes it much, much easier to find the x-interecepts (which I'll talk about in a bit). For example...

Standard form: $f(x) = x^2 + x - 6$
Factored form: $f(x) = \underbrace{(x-2)}_{\text{factor}}\underbrace{(x+3)}_{\text{factor}}$

Standard form: $f(x) = 5x^2 + 5x$
Factored form: $f(x) = 5x(x+1)$

On the other hand, consider something like $f(x) = (x-2)(x+3)^2 - 7$. This is neither factored nor in standard form. It's just a weird Frankenstein polynomial.

[3]I mentioned earlier that if you could write polynomials with infinitely many terms, you could write exponential functions and trig functions as polynomials. Here are two common examples:

$$e^x = \sum_{k=0}^{k=\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = \sum_{k=0}^{k=\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

(Remember that the exclamation mark is the symbol for the factorial, e.g., $4! = 4 \cdot 3 \cdot 2 \cdot 1$.) Look up "Taylor series" if you want to learn more.

Factoring can be tough. When you first learn it, it's really hard. There's no formula for how to do it. You're sitting there, a wee little eighth-grader confronting quadratics like $x^2 - 2x - 15$, and you're not sure what to do. So you write down $(x+\quad)(x+\quad)$, and stare at it for a while. You experiment. You try some numbers, multiply them out, and they don't work. So you try some more. And eventually—in a blinding flash of insight—you realize how to do it! You realize that it's $(x+3)(x-5)$! In a sense, then, this is the first time that you really get to experience the *feeling* of doing math. You get that feeling of being lost in the woods without a map, and having to stumble through, groping at dark branches, trusting that eventually you'll make it out, and then finally, without warning, coming to an opening and pushing the branches aside and seeing the glory of a beautiful [something]!

You keep factoring. You do more and more problems, and it gets easier. You come up with little strategies for how to factor, little mental heuristics, and you develop an intuition. And you wonder: could you formalize this intuition? Meaning: you've gotten good at factoring, but you still don't really want to have to factor every quadratic you see by hand. It would be nice if you could do it automatically. It would be nice if you could come up with an equation that tells you how to factor!

So you work on this project for a while. And eventually you come up with the method of completing the square, and eventually you derive the quadratic equation. And with the quadratic equation, you can now factor every quadratic!!! To wit:

$$ax^2 + bx + c = \left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

If you're skeptical, just multiply out the right side (using FOIL or something). After a lot of simplification, you'll get just $ax^2 + bx + c$! Isn't it beautiful? Here's a picture of the quadratic equation spray-painted onto a Jersey barrier in New York[4]:

It turns out that we can do the same thing for any cubic polynomial. Factor it, I mean. Though I guess you could also use it as a graffiti subject. We can come up with a "cubic formula" which is hideously long but does factor any third-degree polynomial. Likewise with quartics (fourth-degree polynomials), though the quartic formula is so long and messy that it's kind of a nightmare. But as it turns out, for fifth-degree polynomials and above, there is no general formula for factoring... i.e., sometimes it's *impossible* to factor fifth-degree and greater polynomials! Thank Évariste Galois (1811-1832) for proving that, and then getting killed in a duel over a girl before he could do even more awesome mathematics.

[4]reprinted under a Creative Commons license from http://www.flickr.com/photos/37665276@N00/76780093

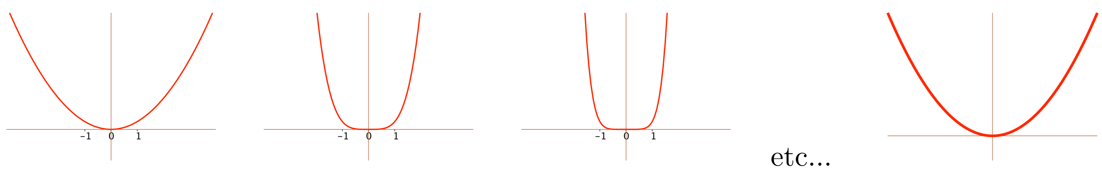
Honestly, although it might seem boring on this level, the theory of factoring is one of my favorite things in mathematics. When you do it abstractly enough (i.e., you define a polynomial such that it might not involve numbers, or even arithmetic), there is a lot of beauty in factorization. It requires a lot of mathematical machinery, but there are some really amazing connections between whether you can factor a polynomial and the symmetry of the coefficients. I don't really know how to explain it in a way that would make any sense to someone who hasn't seen a lot more math, but it really is quite beautiful.

Anyway, take abstract algebra before you die. Hopefully a lot sooner. Getting back on topic, the one disadvantage to writing a polynomial in factored form (aside from the fact that it's difficult and sometimes impossible) is that it's not as immediately obvious what the degree is. But it's not hard to figure out—just count up all the x's (including the exponents on the outside of factors). For example, the function $f(x) = x(x-2)(x+4)^3(2x-6.39)^2$ is of degree 7.

More broadly, **the reason we care about factoring** is that we want to sketch polynomials, and so we need to know the x-intercepts. If we know the factored form of a polynomial, it's very easy to find the x-intercepts. Which brings us to the graphs of polynomials.

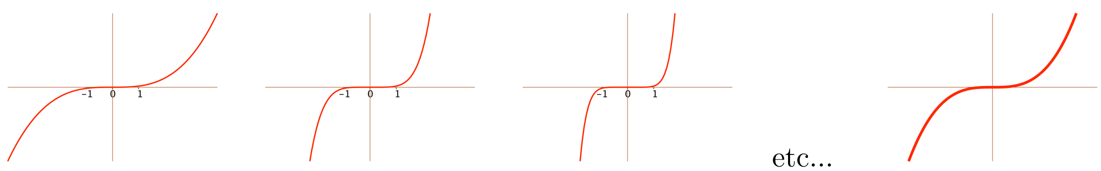
Graphs of Polynomials

You probably already know that simple, one-term polynomials ("monomials") are quite predictable. Even-degree monomials look like this:



$f(x) = x^2$ $\quad$ $f(x) = x^4$ $\quad$ $f(x) = x^6$ $\quad$ etc... $\quad$ $f(x) = x^n$, n even

And odd-degree monomials look like this:



$f(x) = x^3$ $\quad$ $f(x) = x^5$ $\quad$ $f(x) = x^7$ $\quad$ etc... $\quad$ $f(x) = x^n$, n odd

But for the most part, we are curious about polynomials that have lots of terms—stuff like

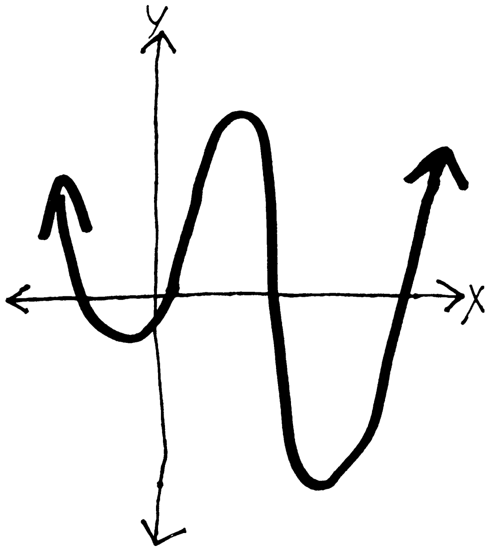
$$f(x) = 4x^6 - 20x^5 + 20x^4 + 3x^3 - 5$$

What happens when we add all of those extra terms? How do all of those cross-terms affect the shape of the polynomial? The basic idea is that when we add these additional terms, we might introduce the possiblity of factoring the polynomial differently, which would give us new x-intercepts, and consequently, new maxima and minima. But the polynomial still retains the "general shape" of its leading term—meaning that if you zoom out far enough, the polynomial above will look more and more like

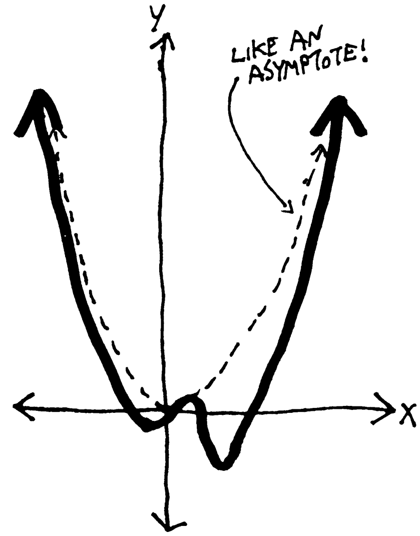
$$f(x) = 4x^6$$

In other words, it'll look more and more like a parabola. Here's what the function looks like:

zoomed in



zoomed out



Another way of thinking about this is that the polynomial has an *end asymptote* at whatever its leading term is. In the above example, for instance, the polynomial $f(x) = 4x^6 - 20x^5 + 20x^4 + 3x^3 - 5$ has an end asymptote at $f(x) = 4x^6$ (a parabolic asymptote).

Let me see if I can try to justify the fact that the end behavior depends only on the leading term. This isn't a formal proof, but it does contain the basic idea as to why this behavior happens. Imagine you have some polynomial. Say you have $f(x) = x^6 + x^4 + 3x^2 + 7x + 24$. What happens as you plug in bigger and bigger values for x? Let's make a table and see.

x	x^6	x^4	$3x^2$	$7x$	24
0	0	0	0	0	24
1	1	1	3	7	24
5	15,625	625	75	35	24
10	100,000	1,000	300	70	24
100	1,000,000,000,000	100,000,000	30,000	700	24
1000	1,000,000,000,000,000,000	1,000,000,000,000	3,000,000	7,000	24

What's going on here? When we plug in zero for x, everything is zero, except for the constant term. Near $x = 0$, all the other terms will be comparatively small, so the constant term (to use a fancy phrase) is the "dominant term", and the function will be close to $y = 24$. But as we plug in larger values for x, 24 just stays the same, and all of a sudden other terms start getting larger than it.

As we plug in bigger and bigger values for x, not only is the leading term (x^6) getting exponentially bigger, but the *difference* between x^6 and all the other terms is *itself* getting exponentially bigger. Compared to 24, 7,000, or even one trillion, 1,000,000,000,000,000,000 is just way, way bigger. (It's 100,000 times larger than even 1,000,000,000,000.)[5]

Let's look at that a little more closely. Here's the table again, but this time we'll include a column that measures the difference between the leading term (x^6) and all the other terms (x^4, $3x^2$, $7x$, 24). We'll do that by writing (in the far-right column) the leading term as a percentage of all the other terms—i.e., we'll compute $\dfrac{x^6}{x^4 + 3x^2 + 7x + 24}$.

[5]You can see these more easily if you write them in scientific notation: $1,000,000,000,000,000,000 = 10^{18}$ and $1,000,000,000,000 = 10^{12}$. So then $10^{18}/10^{12} = 10^{18-12} = 10^6 = 100,000$.

x	x^6	x^4	$3x^2$	$7x$	24	leading term as a % of other terms
0	0	0	0	0	24	0%
1	1	1	3	7	24	2.8%
5	15,625	625	75	35	24	610%
10	100,000	1,000	300	70	24	2,495%
100	1,000,000,000,000	100,000,000	30,000	700	24	250,000%
1000	1,000,000,000,000,000,000	1,000,000,000,000	3,000,000	7,000	24	25,000,000%

Do you see that? By the time x is 5, the leading term is six times bigger than all the other terms *combined*. By the time we're at $x = 100$, it's 2,500 times bigger.

In summary, then, it's easy to find out what a polynomial looks like on the far left side and the far right side—we just look at the leading term. The tails of a polynomial always go either up to infinity or down to negative infinity—there are never any horizontal asymptotes or anything like that. Put differently...

- **Polynomials of even degree**: tails both go in the same direction.
  - Leading coefficient $+$: tails both go up (like $+x^2$).
  - Leading coefficient $-$: tails both go down (like $-x^2$).
- **Polynomials of odd degree**: tails go in opposite directions.
  - Leading coefficient $+$: like $+x^3$
  - Leading coefficient $-$: like $-x^3$.

The real question, then, is what happens in the *middle* of the graph. In the middle of the graph (i.e., near the origin), polynomials bounce up and down and up and down and etc. etc. etc.—this is why we love polynomials—and they always do so smoothly and continuously, with never a vertical asymptote (like $1/x$ has) nor cusp (like $\sqrt[3]{x^2}$ has) or anything messy like that. Polynomials are very nice functions. They are continuous everywhere and differentiable everywhere.

Here are some fun facts (proofs omitted—you can do them with calculus, though).

- A polynomial of degree n has
  - at most n x-intercepts (maybe fewer)
  - at most $n - 1$ extrema (maxima and minima) (but again, it might have fewer)
  - at most $n - 2$ inflection points (points where the concavity changes, and yes, it can have fewer than $n - 2$)

And here's what we really want to talk about, so that we can know how these polynomials bounce up and down:

x-Intercepts/Roots/Solutions/Zeroes

- Four different names for the same thing—basically interchangeable. I usually use "root" or "zero", but it doesn't really matter. Mad important! Algebraic geometry! Roots!
- Occur wherever the polynomial = 0 (given a polynomial $f(x)$, set $f(x) = 0$ and see what values of x make that true.)
- We already said that polynomial of degree n can have up to n roots (can have fewer).
  - Even-degree polynomials can have as few as 0 roots (why? Give an example.)

- Odd-degree polynomials must have at least 1 root (why?)

- Easy to find x-intercepts when polynomial is in factored form. When any individual factor = 0, the whole function will be 0 (because $0 \cdot (\text{stuff}) = 0$)

- So it's sufficient just to find where each factor equals zero (i.e., each factor has a corresponding x-intercept/root/solution/zero).
  - For example, consider the polynomial $f(x) = (x+2)(x-3)$. We know the polynomial will have roots wherever $(x+2) = 0$, and wherever $(x-3) = 0$. So it has roots at $x = -2$ (from the first factor) and $x = 3$ (from the second factor).
  - Note that some factors might not cause x-intercepts—for example, consider (x^2+1). If we set this equal to zero and try to solve for x, we get $x = \sqrt{-1}$, which is not a real number (it's a complex number). So it doesn't show up on the graph, and doesn't affect any of the main properties of the graph (except maybe the y-intercept).

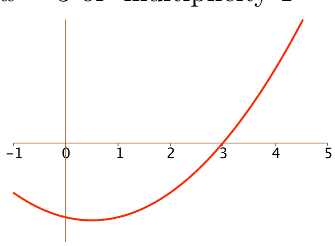
- Each root also has a **multiplicity**. We say the **multiplicity** of an x-intercept/solution/zero/root is the number of times the factor creating the particular root shows up in the complete factorization (in the factored form of the polynomial).

  For example, the polynomial $f(x) = (x-3)(x+5)^2$ has a root at $x = 3$ of multiplicity 1, and a root at $x = -5$ of multiplicity 2.

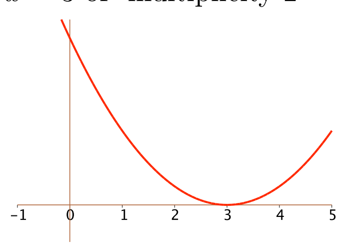
  Why do we care? Because the multiplicity tells us what the polynomial looks like near that root! it tells us the shape of the polynomial near the x-intercept/root/solution/zero

  In particular...
  - if the root is of multiplicity 1: polynomial looks like a straight line near that root (crosses through axis)
  - if the root is of even multiplicity: polynomial looks like a parabola (even-degree polynomial) near the root (i.e., it only bounces off the axis; it doesn't cross it).
  - if the root is of odd multiplicity: polynomial looks like an odd-degree polynomial near the root (e.g., crosses through axis and is bendy as it goes through).

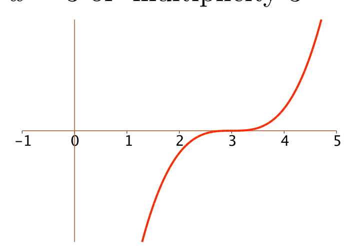
This fxn has a root at $x = 3$ of multiplicity 1



This fxn has a root at $x = 3$ of multiplicity 2



This fxn has a root at $x = 3$ of multiplicity 3



y-intercept

- This can also be useful. Since the y-axis is just the line $x = 0$, the y-intercept is just the value of the function when $x = 0$. So we can plug 0 in for x to find the y-intercept.

- For example, consider the polynomial $f(x) = 5x^6 - 2x^3 + 4x + 32$. Then $f(0) = 5 \cdot 0^6 - 2 \cdot 0^3 + 4 \cdot 0 + 32 = -12$, so the y-intercept is at $y = 32$. Which is just the constant term.

- Or if we have something factored, like $f(x) = (x+3)(x-4)(x+2)^2$. Then $f(0) = (0+3)(0-4)(0+2)^2 = (3)(-4)(2)^2 = -48$. So the y-intercept is at $y = -48$.

Let's graph a real polynomial!

- Consider a complicated example: $f(x) = x(x-2)(x+4)^3(2x-6.39)^2$. Let's graph it!
- First, let's find the x-intercepts/roots/solutions/zeroes. This function has four factors, so we'll set each of them equal to zero and figure out the corresponding x-intercept/root/solution/zero.
  - if $x = 0$, then $x = 0$ is an x-intercept/etc. (of multiplicity 1)
  - if $x - 2 = 0$, then $x = 2$ is an x-intercept (of multiplicity 1)
  - if $(x+4)^3 = 0$, then we do some more algebra...

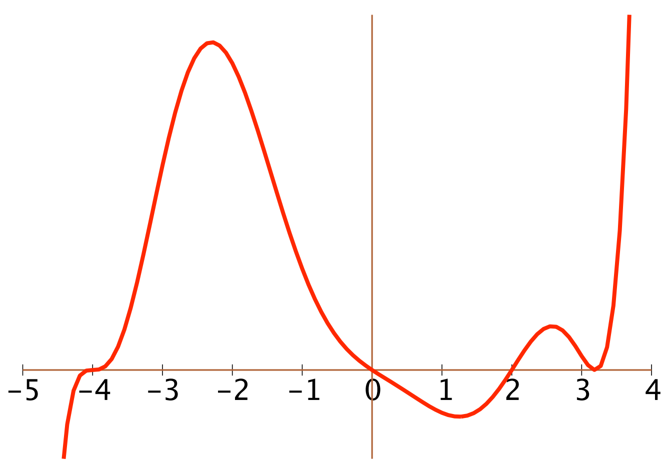
$$\begin{aligned} (x+4)^3 &= 0 \\ \sqrt[3]{(x+4)^3} &= \sqrt[3]{0} \\ (x+4) &= 0 \\ x &= -4 \end{aligned}$$

    so $x = -4$ is an x-intercept. And, since the factor $(x+4)^3$ is cubed, the root is of multiplicity 3.
  - and likewise with $(2x-6.39)^2$, which gives us an x-intercept at $x = 3.195$ (of multiplicity 2).
- We can summarize our findings:

$$f(x) = x(x-2)(x+4)^3(2x-6.39)^2$$

Factor	x	$x-2$	$(x+4)^3$	$(2x-6.39)^2$
Root	0	2	-4	3.195
Multiplicity	1	1	3	2

- How about the y-intercept? We'll just plug 0 in for x, and get: $f(0) = 0(0-2)(0+4)^3(2\cdot 0-6.39)^2 = 0$. So this polynomial has a y-intercept at 0
- Also, we know that the polynomial is 7th degree, and the leading term is positive, since there were no negatives in front of any of the x's in the factorization (and no negative in front), so the general shape will look like $+x^3$.
- So now we can graph it! All we have to do is plot the x-intercepts and, basically, connect the dots—we need to know where to start drawing a line from (in this case, starting at the bottom left and ending at the top right, because it looks like $+x^3$), and we need to know what the polynomial looks like at each of the x-intercepts (which the multiplicity tells us). But then we can do it!



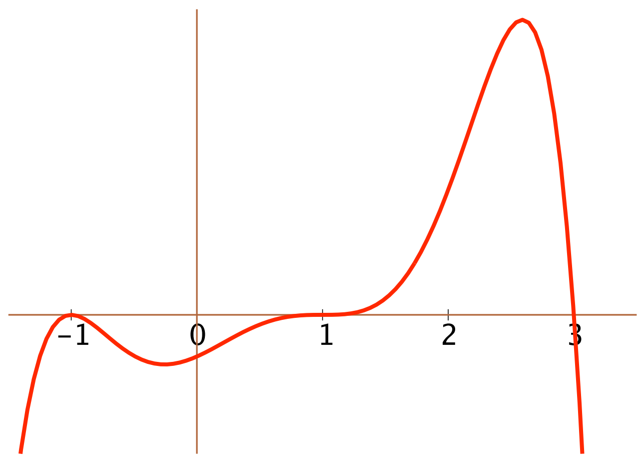
Another example

- Consider $f(x) = -(x-3)(x+1)^2(x-1)^3$
- This is sixth-degree, and there's a negative in front, so the general shape is like $-x^2$ (so both the tails go down).
- It has a y-intercept at $f(0) = -(0-3)(0+1)^2(0-1)^3 = -3$.
- And the roots and their multiplicities are thus:

$$f(x) = -(x-3)(x+1)^2(x-1)^3$$

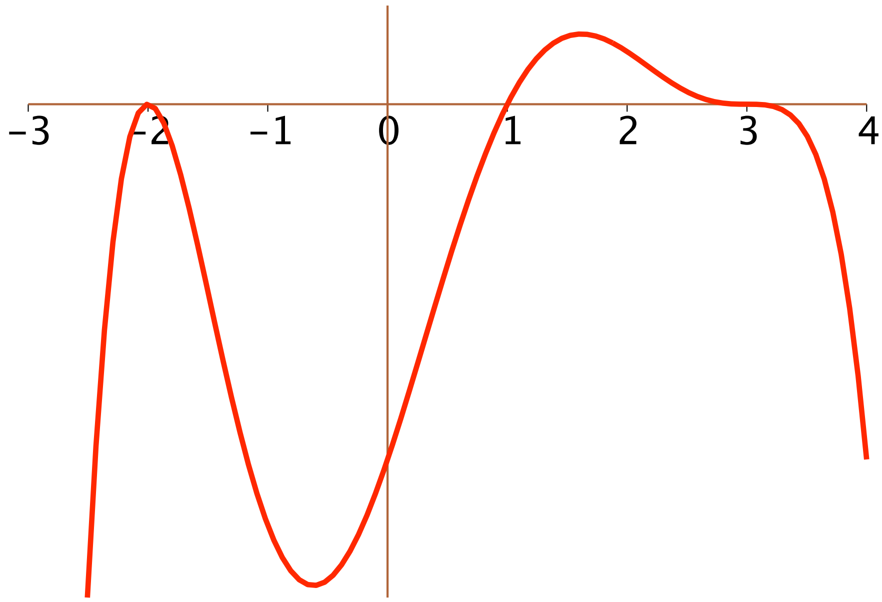
Factor	$(x-3)$	$(x+1)^2$	$(x-1)^3$
Root	3	-1	1
Multiplicity	1	2	3

- So our sketch should look like:



Going backwards

- What if we wanted to do the reverse—to walk into the Metropolitan Museum of Art, see a beautiful oil painting of a polynomial on the wall (there is a little-known mathematics gallery underneath the Temple of Dendur), and show off our math skills to our attractive date by saying, "Aha! I can come up with an equation for that polynomial!"?
- We could do just that.
- So imagine that the painting that strikes your fancy looks like this:



- What can you say about this just from looking at it? We can see that both tails point down, so it must be of even degree, and it must have a negative somewhere (making it open down rather than up).

- We can also see that it has three roots.

  - It has a root at $x = -2$, and since the polynomial looks like a parabola near $x = -2$, this root must be of even multiplicity. It might be of multiplicity 2, multiplicity 4, 6, etc. So a possible factor of the polynomial could be $(x + 2)^2$.
  - It has a root at $x = +1$, and since it goes through the axis, straight through, that root must be of multiplicity 1. So it must be created by the factor $(x - 1)$.
  - Finally, it has a root at $x = +3$, and since it goes through the axis and is kinda bendy as it goes through, that root must be of odd multiplicity—maybe multiplicity 3, maybe 5, maybe 341. So a possible factor could be $(x - 3)^3$.

  We can summarize our findings, similarly to how we did before:

Root	$x = -2$	$x = +1$	$x = 3$
Multiplicity	even	1	odd
Possible Factor	$(x + 2)^2$	$(x + 1)$	$(x - 3)^3$

- So now we can take a stab at coming up with an equation! We'll just pile all those factors together, and remember to include a $-$ in front:

  $f(x) = -(x + 2)^2(x + 1)(x - 3)^3$

- Could this be an equation for this graph? Sure. The polynomial we've written is of degree 6 (so it's even, which checks out). It has a negative in front, which will make it open downward (good). And, finally, we can check the y-intercept: $f(0) = -(0 + 2)^2(0 + 1)(0 - 3)^3 = -(1)^2(1)(-3)^3 = -108$. The y-axis on our graph doesn't have a scale, but we can see that the y-intercept should be some negative number—so this appears to work. Hooray!

- So you tell this to your date. And she or he says, "Mmm... but couldn't it also be $f(x) = -(x + 2)^4(x + 1)(x - 3)^7$?" And you stammer, "Well, of course. That function would still have the same basic properties—it just might have a different vertical scale, and some of the roots might be a little more shapely."

Then she or he says, "What about $f(x) = -(x+2)^4(x+1)(x-3)^7(x^2+1)$?"

And you say, "Of course not! What's that extra (x^2+1) factor doing in there? There's no root at... oh."

She says, "It works. The factor (x^2+1) doesn't give you a real-valued root that you can plot on the x, y plane. If you try to solve it, you get:

$$(x^2+1) = 0$$
$$x^2 = -1$$
$$x = \pm\sqrt{-1}$$

"The square root of negative one is a complex number, not a real one. It'll change the vertical scale of the polynomial—it's like a non-linear vertical expansion, if you can imagine that—but it'll still have all these basic properties."

And then she walks off toward the Henry Moore exhibit.

Problems

For the following polynomials:

1. What is the degree of the polynomial?
2. What is the sign of the leading coefficient?
3. How many real roots/solutions/zeroes/x-intercepts does it have? Where are they, and what are their multiplicities?
4. What is the y-intercept?
5. What does the polynomial look like? (i.e., sketch it. Without a calculator!)

(You might have to factor or otherwise manipulate the expressions to get them into a convenient form.)

1. $f(x) = (x-2)(x+4)$

2. $h(x) = x^2 + 8x + 15$

3. $t(x) = x^2 - 2x - 15$

4. $r(x) = x^2 - x - 2$

5. $w(x) = x^2 - 2x - 63$

6. $f(x) = x^3 + x^2 - 4x - 4$

7. $j(x) = x^4 + 2x^3 + 9x + 18$

8. $g(x) = x^5 - x^3 + 5x^2 - 5$

9. $h(x) = (x-2)(x+3)(x+5)$

10. $t(x) = -(x-1)(x+9)(x-7)(x+3)$

11. $r(x) = (x+4)(x-2)(x+6)(x-5)$

12. $f(x) = 5(x-2)(x+10)(x+4)$

13. $f(x) = 3x^2(x-1)(x+7)(x+8)$

14. $w(x) = (x-6)(x+4)^2(x-1)^3$

15. $f(x) = (x+3)^2(x+5)^2$

16. $j(x) = (x+1)^2$

17. $f(x) = (x+1)(x-5)^2(x^2+3)$

18. $f(x) = -(x^2-1)(x-3)^2x^2$

19. $f(x) = 5x(3x+2)(x-5)(x+6)$

20. $f(x) = (x^3+27)(x+2)^2$

21. $f(x) = -(x^2-x-12)(x^2+8x+12)$

22. $f(x) = (x+1)^2(x+7)^7(x^2+3)$

23. $f(x) = x^3(x^2-9)$

24. $f(x) = (x-a)(x+5a)^2(x+a)$, where a is a constant greater than zero.

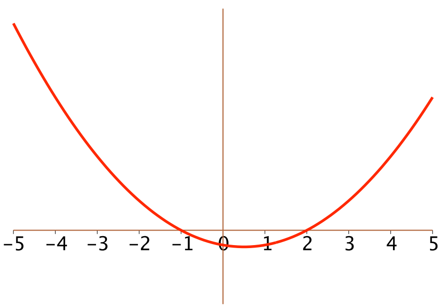
25. $f(x) = a^2(x-3a)(x+a)^3$, where a is a constant greater than zero.

26. $f(x) = (x-a)(x+b)^2(x+5b)$

For the following graphs of polynomials:

1. What is the degree of the polynomial (even or odd)?
2. What is the sign of the leading coefficient (positive or negative)?
3. How many real roots/solutions/zeroes/x-intercepts does it have? Where are they, and what are their multiplicities?
4. How many extrema (minima and maxima) does the polynomial have?
5. What could a possible equation for the polynomial be? (Give a factorized form.)
6. What could a *second* possible equation for this polynomial be? (i.e., an equation which gives a polynomial that has the same general shape and the same x-intercepts.)

(Note that the x-axes and y-axes of these graphs have different scales. Also note that the number of the problem is at the *bottom*-left corner of the graph (not the top).)



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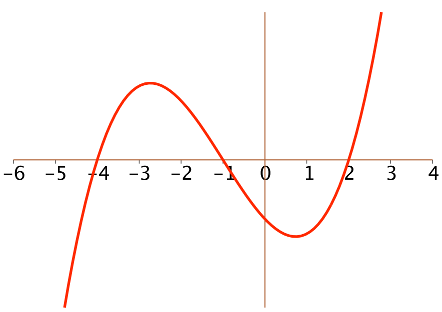

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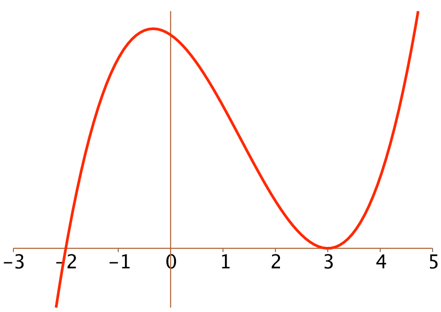

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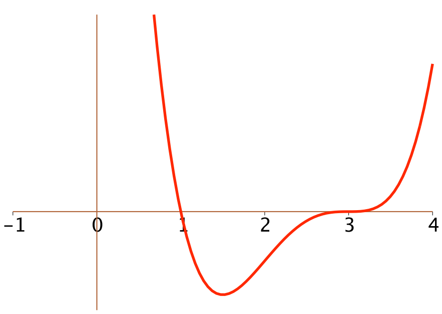

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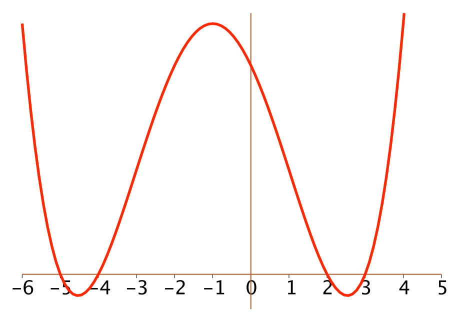

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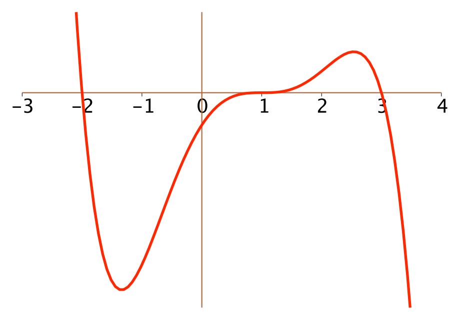

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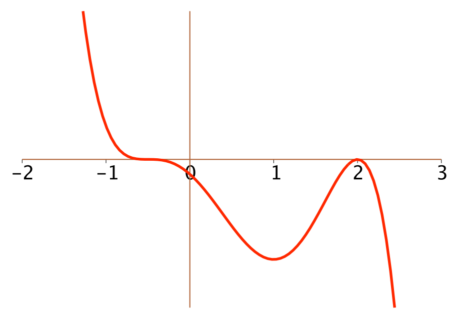

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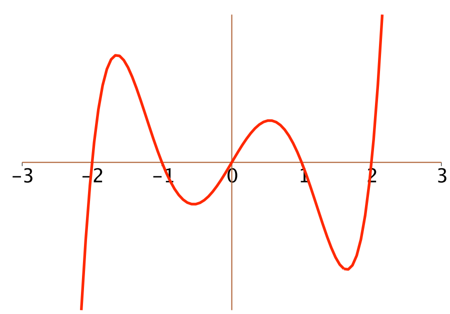

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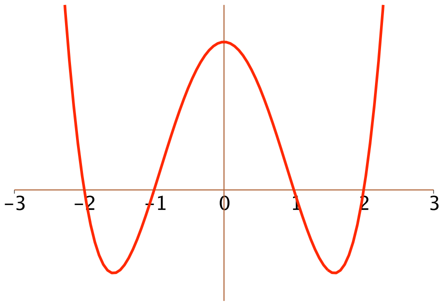

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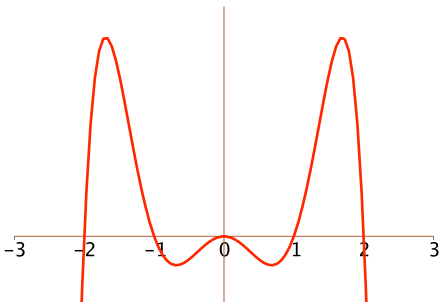

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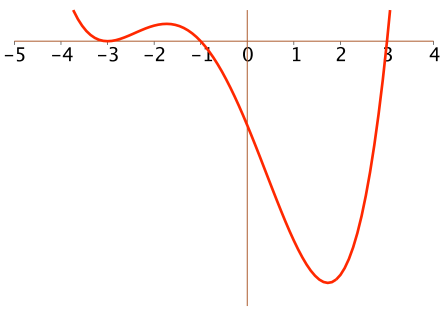

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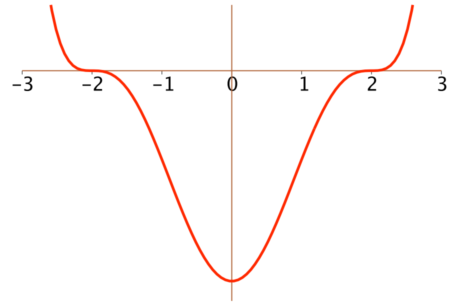

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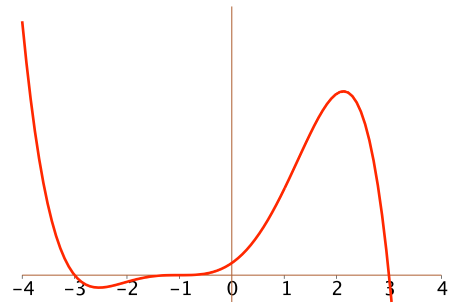

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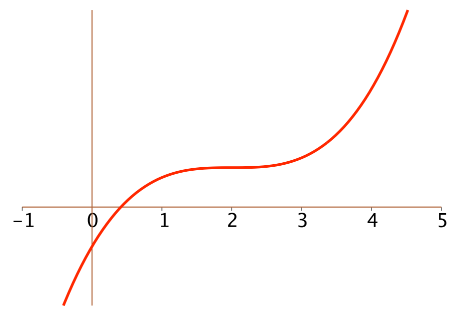

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