Two Additional Complex Numbers Problems

Math 3

These were two of the problems I was going to write for our complex numbers test, tragically canceled due to COVID-19. Think about them, try to solve them, write down what you figure out, and post it to Canvas.

Two questions arising from our months-ago calculation of the cube roots of i in rectangular form.

Cole's Conjecture: Cole was very excited to find that the (fraternal!) third cube root of i is just $-i$. He hypothesized that because of the cyclic nature of the powers of i, any multiple-of-three'th root of i will *also* have $-i$ as one of the possibilities. Was he correct? Prove or disprove. Clearly there are *some* multiple-of-three'th roots of i that have $-i$ as a possibility (the cube root). Do all of them? Is there a similar true conjecture involving which roots of i will have $-i$ as a solution?

Maya's Musing: Maya found two of the roots, $\pm\frac{\sqrt{3}}{2} + \frac{1}{2}i$ in the straightforward ordinary way. She found the third root, $-i$, though, in a more novel method: by assuming that all three of the roots have to multiply together to create i:

$$\text{(the first root)} \cdot \text{(the second root)} \cdot \text{(the unknown third root)} = i$$

And then assuming that the unknown root is another complex number in the form $a + bi$, giving this equation:

$$\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \cdot \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \cdot (a + bi) = i$$

And then solving for a and b. And indeed, she calculated that the third root should be $-i$!

Muse on this method. It worked in this case—will it always work? Clearly, if we take an n'th root of z, and multiply it by itself n times, we get z. That's what a root is. But is the product of *all the distinct* n'th roots of z, together always equal to z? Prove or disprove. Can you find a counterexample? If it's not always equal to z, is it sometimes equal to z? When?