Complex Roots, Explained, Somewhat

Math 3

My wee cousins Olive and Leo, until their weights exceeded my upper-body strength, used to come up to me and shout "SPIN ME!!!!!" I'd pick them up by the crook of their shoulders, spin them around, their legs and torso would fly up perpendicular to the ground due to conservation of angular momentum (or something), I'd have to lean back slightly (so our combined center of mass was still over my feet), and I'd keep spinning them until I ran out of glycogen. Unbeknownst to Olive and Leo, not only were they demonstrating several important principles of kinematics and rotational dynamics, they were also demonstrating some important properties of complex exponentiation (or rather, complex *de*-exponentiation, a/k/a roots).

Let's back up a bit. You spent several days in class working out problems from the worksheet titled $\sqrt{\text{Complex!}}$. The problems all increased in difficulty and generality, concluding (hopefully) with you working out a formula for all of the nth roots of any complex number. Hopefully you found something like this:

$$\sqrt[n]{r\angle\theta} = \left(r^{\frac{1}{n}}\right)\angle\left(\frac{\theta+2k\pi}{n}\right) \qquad \text{for } k \in \{0,1,2,\ldots,n-1\}$$

You might have written it in a different form, or with different variable or notational choices; that's fine.

In a sense, this is what most of our work so far this semester has been building to. We started the semester, on the very first day, by squaring these two expressions to prove that they're apparently both the square roots of i:

$$\left(\pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i\right)^2 = i$$

But that was unsatisfying, since I just pulled those expressions out of a hat. Sure, if we square them, we get i, proving that they're both square roots of i. But that doesn't tell us where they came from. That doesn't tell us *why* they're square roots of i.

Then we learned how to find those two square roots algebraically, by setting up an equation and solving it:

$$\sqrt{i} = a + bi$$

That gave us a *procedure* for finding the square roots, which was a little bit better. At least we weren't pulling things out of a hat. And then by using that procedure, we were able to also find the cube roots and quartic roots of i:

$$\text{to find the cube roots of } i\text{, we solved: } \sqrt[3]{i} = a + bi$$

$$\text{to find the quartic roots of } i\text{, we solved: } \sqrt[4]{i} = a + bi$$

But those procedures involved increasingly tedious and untenable amounts of algebra. Expanding things to the fourth power? Egads! Plus we got all these weird, nasty answers. They didn't make any intuitive sense.

And then we realized that complex numbers have this amazing geometric behavior! We realized that if we multiply two complex numbers together, their radii multiply and their angles add. By extension, we realized that if we exponentiate a complex number, its radius gets exponentiated, and its angle gets multiplied by the exponent. Then we created polar coordinates as a way to take advantage of these observations. With polar coordinates, and our newfound geometric understanding of complex numbers, we were able to understand why complex numbers have the roots they do. That's what you figured out (I hope) on that worksheet.

I tried to summarize this geometric understanding of the cube roots of i with this drawing I did in class. We saw each of the three cube roots of i, each spinning around the origin different amounts, but all ending up at $i = 90°$:

PICTURE

But this was sort of easy, because we already knew what the cube roots of i were (from our many calculations of them). More generally, how can we think about finding the nth roots of any complex number, intuitively?

We'll use Olive and Leo. Suppose we're trying to find all the fourth roots of $1\angle 120^\circ = e^{2i\pi/3}$. Let's think about all the ways we can spin Olive and Leo, starting at $1\angle 0^\circ$, and dropping them at $1\angle 120^\circ$:

- We can spin around the unit circle **zero** times, plus an additional 120°, to get to $1\angle 120^\circ$
- We can spin around the unit circle **once**, plus an additional 120°, to get to $1\angle 120^\circ$
- We can spin around the unit circle **twice**, plus an additional 120°, to get to $1\angle 120^\circ$
- We can spin around the unit circle **three** times, plus an additional 120°, to get to $1\angle 120^\circ$
- We can spin around the unit circle **four** times, plus an additional 120°, to get to $1\angle 120^\circ$
- We can spin around the unit circle **five** times, plus an additional 120°, to get to $1\angle 120^\circ$
- We can spin around the unit circle **six** times, plus an additional 120°, to get to $1\angle 120^\circ$
- etc. ("KEEP GOING KEEP GOING KEEP SPINNING ME KEEP SPINNING ME")

How much do we spin, in total, each time?

- We can spin $0^\circ + 120^\circ = 120^\circ$ to get to $1\angle 120^\circ$
- We can spin $360^\circ + 120^\circ = 480^\circ$ to get to $1\angle 120^\circ$
- We can spin $720^\circ + 120^\circ = 840^\circ$ to get to $1\angle 120^\circ$
- We can spin $1080^\circ + 120^\circ = 1200^\circ$ to get to $1\angle 120^\circ$
- We can spin $1440^\circ + 120^\circ = 1560^\circ$ to get to $1\angle 120^\circ$
  ("ANDREW PWEEEZ DONT DROP ME!!!")
- We can spin $1800^\circ + 120^\circ = 1920^\circ$ to get to $1\angle 120^\circ$
- We can spin $2160^\circ + 120^\circ = 2280^\circ$ to get to $1\angle 120^\circ$
- etc. (keep going until you get tired)

So, we've got all these different ways of making $1\angle 120^\circ$! If we're trying to find all its fourth roots, we need a number that, when raised to the four, gives us $1\angle 120^\circ$. The radius will just be 1, since $1^4 = 1$. But what about the angles? When we multiply complex numbers, the angles add. When we exponentiate complex numbers, the angles multiply. So to *de*-exponentiate complex numbers (i.e., take roots), we need to divide the angles. In this case, to find the fourth roots, we need angles that, when multiplied by four, get us to the final angle. So we need:

- If we spin a total of 120°, the corresponding fourth root must have an angle of $120/4 = 30^\circ$
- If we spin a total of 480°, the corresponding fourth root must have an angle of $480/4 = 120^\circ$
- If we spin a total of 840°, the corresponding fourth root must have an angle of $840/4 = 210^\circ$
- If we spin a total of 1200°, the corresponding fourth root must have an angle of $1200/4 = 300^\circ$
- If we spin a total of 1560°, the corresponding fourth root must have an angle of $1560/4 = 390^\circ \cong 30^\circ$
- If we spin a total of 1920°, the corresponding fourth root must have an angle of $1920/4 = 480^\circ \cong 120^\circ$

- If we spin a total of 2280°, the corresponding fourth root must have an angle of $2280/4 = 570° \cong 210°$
- etc.—but they're repeating!

So, thanks to spinning around more and more times, we can find all the fourth roots! After we spin around four total rotations (plus the little bit extra to get to 120°), the roots start repeating themselves. So we've found all four distinct fourth roots of $1\angle 120°$:

$$\begin{aligned}\sqrt[4]{1\angle 120°} = & 1\angle 30°, \quad \text{(from spinning zero times)} \\ & 1\angle 120°, \quad \text{(from spinning once)} \\ & 1\angle 210°, \quad \text{(from spinning twice)} \\ & 1\angle 300°. \quad \text{(from spinning thrice)}\end{aligned}$$

Or, in exponential/Euler form:

$$\begin{aligned}\sqrt[4]{e^{2i\pi/3}} = & e^{i\pi/6}, \quad \text{(from spinning zero times)} \\ & e^{2i\pi/3}, \quad \text{(from spinning once)} \\ & e^{7i\pi/6}, \quad \text{(from spinning twice)} \\ & e^{5i\pi/3}. \quad \text{(from spinning thrice)}\end{aligned}$$

(By the way, you may note the cool coincidence that $1\angle 120°$ is its own fourth root! Why is that?) Somewhat more succinctly, we can describe all the fourth roots of $e^{2i\pi/3}$ thusly:

$$\begin{aligned}\sqrt[4]{1\angle 120°} &= 1\angle\frac{120+360k}{4} \qquad \text{for } k \in \mathbb{Z} \text{ (but with repeats)} \\ &= 1\angle\frac{120+360k}{4} \qquad \text{for } k \in \{0,1,2,3\} \text{ (without repeats)}\end{aligned}$$

By the way, I've been a little casual with the formalism here—I've been slipping into different representations of complex numbers in these notes, as well as going back and forth between radians and degrees. I don't think I've introduced any ambiguities (eep), but please do let me know if something is unclear, or if you think I've made a mistake.

More generally, we have:

$$\sqrt[n]{r\angle\theta} = \left(r^{\frac{1}{n}}\right)\angle\left(\frac{\theta + k\cdot 2\pi}{n}\right), \qquad k \in \mathbb{Z}$$

By the way, a couple of you have complained about the radius, and worried that your formulas were erroneously recursive. We're trying to find the nth root of $r\angle\theta$, and our formula has an nth root of r in it! That's a very valid complaint. Here's my response: by "$r^{\frac{1}{n}}$" in that formula, I mean "the normal, ordinary, positive real root." Radii are always positive real numbers—there's nothing messy about them. The fancy name, by the way, for "the normal boring positive real root" of some complex number is its **principal root**.[1]

What about k? If we let it be any integer, then we'll have an infinite number of roots, but because we're dividing by n, there will only be n distinct roots. Those n distinct roots with the smallest total-amount-of-spinning usually get called **primitive roots**, as opposed to the roots with way too much spinning. So in some sense, any number has an infinite number of nth roots, but it only has one principal root, and only n primitive roots. But of course these are just words people use.

[1] Or, uh, okay, I think the more common definition is "the root with smallest angle," since if we're taking a root of a nonreal number, it won't necessarily have a real principal root. But here we're taking a root of the radius; the radius is always a positive real; so it always has a real root.