Complex roots, in a Cartesian sense

Math 3

So we figured out how to take the square root of i. We did this by assuming that the square root of i was *some* complex number, and thus we could write it in the form $a + bi$, and thus we got an equation, and solved it for a and b:

$$\sqrt{i} = a + bi$$

$$\vdots$$

lots of algebra later...

$$\vdots$$

$$= \frac{\pm 1}{\sqrt{2}} + \frac{\pm 1}{\sqrt{2}} i$$

This method seemed to work, but the result is weird. We still don't have a good understanding of what the square root of a complex number *is*. I mean, by definition, a square root of any number is just the number that, when you multiply it by itself, you get the original number back. And that's what we found here. But it's still so *weird*. Where are the fractions coming from? The $\sqrt{2}$? We know *what* it is, but we don't know *why* it is.

Contrast this with taking the square roots of real numbers (positive real numbers). We have a pretty good idea about how that works. For example, we don't know exactly what $\sqrt{17}$ is, but it's definitely something less than 17. Square roots make things smaller (at least when we're talking about real numbers). Or, I guess they actually make them bigger if it's less than 1. So I guess we should say that when we're taking the square root of a real number, that makes it closer to 1.

$$1 < \sqrt{17} < 17$$

And we know more than this. We know $\sqrt{17}$ is a little bigger than $\sqrt{16}$, which is 4, so $\sqrt{17}$ must be a little bigger than 4. And it has to be less than $\sqrt{25}$, which is 5:

$$4 < \sqrt{17} < 5$$

So we've got some basic intuition and ideas for how square roots work with positive real numbers. *But with complex numbers?!?!* No clue. If you're in Block 7, you got to watch that wonderful short documentary on the proof of Fermat's Last Theorem, and Andrew Wiles describes doing mathematics as feeling like:

> Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room, and it's dark, completely dark. One stumbles around bumping into the furniture, and gradually, you learn where each piece of furniture is, and finally, after six months or so, you find the light switch. You turn it on, and suddenly, it's all illuminated. You can see exactly where you were.

Right now, we're stumbling around in the dark. We've bumped into one piece of furniture—the square root of i. Let's bump into some more furniture!

- Can you figure out what $\sqrt[3]{i}$ is, using the same method as we did to find the square root of i? (Quick question beforehand: we found that i has two square roots. This should make sense: 4 also has two square roots, $+2$ and -2. How many cube roots do you think that i has? Analogously, how many cube roots does 8 have?)
- Likewise, can you find the fourth roots of i?
- Graph all this stuff, too.

DON'T PRINT THIS NOT FOR KIDS. Brief notes to myself on finding the cube root of i in rectangular coordinates, as scribbled on Caltrain. We have:

$$\begin{aligned}\sqrt[3]{i} &= a + bi \\ i &= (a+bi)^3 \\ i &= \left(a^3 - 3ab^2\right) + \left(3a^2b - b^3\right) i\end{aligned}$$

(I messed up part of that binomial expansion and got a nightmare cubic, and was worried this was way harder than this was. Thankfully Wolfram Alpha saved me!) So then we get:

$$0 = a^3 - 3ab^2 \quad \text{and} \quad 1 = 3a^2b - b^3$$

Solving the left equation for a gives us:

$$a = 0,\ +\sqrt{3b^2},\ -\sqrt{3b^2}$$

So we have three cases.

- If $a = 0$, then plugging that into the other equation, we get:

  $$1 = 3(0)^2b - b^3$$

  $$b = -1$$

  So that gives us one solution, $\sqrt[3]{i} = 0 - 1i$, or just $-i$.

- If $a = +\sqrt{3b^2}$, we plug that in and get:

  $$1 = 3\left(\sqrt{3b^2}\right)^2 b - b^3$$

  $$b = 1/2$$

  So that gives us $\sqrt{3b^2} + \frac{1}{2}i$. Plugging in $b = 1/2$, this becomes $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ as a solution.

- Finally, if $a = -\sqrt{3b^2}$, we still get the same value for b, and have as a solution $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

So:

$$\sqrt[3]{i} = \left\{-i,\ \frac{\sqrt{3}}{2} + \frac{1}{2}i,\ -\frac{\sqrt{3}}{2} + \frac{1}{2}i\right\}$$

Easy to verify that the first of these works; less pleasant to verify the other two.

What about the quartic root? We have:

$$\begin{aligned}\sqrt[4]{i} &= a + bi \\ i &= (a+bi)^4 \\ i &= \left(a^4 - 6a^2b^2 + b^4\right) + \left(4a^3b - 4ab^3\right) i \quad \text{(Wolfram Alpha did this because I'm lazy)}\end{aligned}$$

So we get:

$$0 = a^4 - 6a^2b^2 + b^4 \quad \text{and} \quad 1 = 4a^3b - 4ab^3$$

Wolfram Alpha says we can factor the left equation and get:

$$0 = \left(a^2 - 2ab - b^2\right)\left(a^2 + 2ab - b^2\right)$$

It's pretty, but eeep. Anyway, then we can... oh, I don't know, maybe this is a good place for the kids just to get lost and frightened! (Presumably we end up with a gross form anyway, since it's $90/4 = 22.5°$, contra the trigonometric cleanliness of $90/3 = 30°$ in the cubic root. Wait, no, now Wolfram Alpha is saying that the roots are just permutations of $\pm\sqrt{2}$ and some other stuff? That's so simple... Hmm, I mean, I guess if we just hit the original equations up with the Q.E. over and over, we can probably get there?)

Okay, I worked this all out on the whiteboard at Nueva. What a total mess. It was basically impossible to keep track of signs, so I totally cheated on that part; were I to rewrite my derivation, I'd more clearly separate the "algebraic simplification of all these fractions of square roots of two" and the "chasing the signs" parts. Also, I started not by expanding $i = (a + bi)^4$, but rather, by taking the square roots of the square roots:

$$\sqrt[4]{i} = \pm\sqrt{\frac{\pm 1}{\sqrt{2}} + \frac{\pm 1}{\sqrt{2}} i} \ = a + bi$$

And solving for a and b thence. Ultimately we get:

$$\sqrt[4]{i} = +\sqrt{\frac{+1+\sqrt{2}}{2\sqrt{2}}} + \sqrt{\frac{-1+\sqrt{2}}{2\sqrt{2}}}\, i, \ -\sqrt{\frac{-1+\sqrt{2}}{2\sqrt{2}}} + \sqrt{\frac{+1+\sqrt{2}}{2\sqrt{2}}}\, i,$$
$$-\sqrt{\frac{+1+\sqrt{2}}{2\sqrt{2}}} - \sqrt{\frac{-1+\sqrt{2}}{2\sqrt{2}}}\, i, \ +\sqrt{\frac{-1+\sqrt{2}}{2\sqrt{2}}} - \sqrt{\frac{+1+\sqrt{2}}{2\sqrt{2}}}\, i$$