Power Sets

Math 12, Veritas Prep.

The **power set** $P(X)$ of a set X is the set of all subsets of X. Defined formally, $P(X) = \{K \mid K \subseteq X\}$ (i.e., the set of all K such that K is a subset of X).

Let me give an example. Suppose it's Thursday night, and I want to go to a movie. And suppose that I know that each of the upper-campus Latin teachers—Sullivan, Joyner, and Pagani—are free that night. So I consider asking one of them if they want to join me. Conceivably, then:

- I could go to the movie by myself (i.e., I could bring along none of the Latin faculty)
- I could go to the movie with Sullivan
- I could go to the movie with Joyner
- I could go to the movie with Pagani
- I could go to the movie with Sullivan and Joyner
- I could go to the movie with Sullivan and Pagani
- I could go to the movie with Pagani and Joyner
- or I could go to the movie with all three of them

Put differently, my choices for whom to go to the movie with make up the following set:

$$\Big\{ \{\}, \{\text{Sullivan}\}, \{\text{Joyner}\}, \{\text{Pagani}\}, \{\text{Sullivan, Joyner}\},$$

$$\{\text{Sullivan, Pagani}\}, \{\text{Pagani, Joyner}\}, \{\text{Sullivan, Joyner, Pagani}\} \Big\}$$

Or if I use my notation for the empty set:

$$\Big\{ \emptyset, \{\text{Sullivan}\}, \{\text{Joyner}\}, \{\text{Pagani}\}, \{\text{Sullivan, Joyner}\},$$

$$\{\text{Sullivan, Pagani}\}, \{\text{Pagani, Joyner}\}, \{\text{Sullivan, Joyner, Pagani}\} \Big\}$$

This set—the set of whom I might bring to the movie—is the power set of the set of Latin faculty. It is the set of all the possible subsets of the set of Latin faculty. (Note that, because of our inclusive definition of a subset[1], a power set always includes the empty set and the original set itself.)

Idle question: what's the power set of the empty set? (Hint: it's not just the empty set. The empty set has two subsets: nothing, and itself. That is, the two subsets of the empty set are 1) the empty set, and 2) the set containing the empty set.) How about the power set of the power set of the empty set (i.e., $P(\,P(\emptyset)\,)$? what about $P(P(P(\emptyset)))$? etc? how many elements do each of these sets have?

Related question: imagine you have a set with n elements (i.e., a set with cardinality n). How many elements does the power set have? (Fancy word for the same thing: what's the **cardinality** of the power set?) Suggestion: try experimenting. Take a set with no elements, and find the cardinality of its power set; take a set with one element and find the cardinality of its power set, take a set with two elements and find the cardinality of its power set, take a set with three elements and find the cardinality of its power set... then, when you can, make a conjecture (an educated guess) about the cardinality of the power set of a set with cardinality n. (To actually prove your conjecture requires some fancy techniques, but that doesn't mean you can't have a decent understanding of why your conjecture is true.)

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[1] $A \subseteq B \iff \forall x(x \in A \Rightarrow x \in B)$, so sets are subsets of themselves. Contrast with our definition of a *proper* subset: $A \subset B \iff \forall x(x \in A \rightarrow x \in B) \wedge \exists y(y \in B \wedge y \notin A)$, i.e., a proper subset is one that actually contains *less* stuff. For example, the reals are a subset of the reals, but the reals are not a proper subset of the reals. The integers, by contrast, are a proper subset of the reals. $\mathbb{R} \subseteq \mathbb{R}$, but $\mathbb{R} \not\subset \mathbb{R}$. On the other hand, $\mathbb{Z} \subseteq \mathbb{R}$, and $\mathbb{Z} \subset \mathbb{R}$. Exactly the same as the distinction, with numbers, between "less than or equal to" and "less than" (hence the notational similarities!).