Inverse Functions

Pre/Calculus 11, Veritas Prep.

One of the most important things we do with functions—I mean, our entire method of calculus is based on functions, so they're pretty important as a structural element, but here I'm talking about functions in the *abstract*—the most important thing we do with functions is *compose* them. We can put two functions together to create a new function. With numbers, we can add two numbers together and get a third number; with functions, we can compose two functions together and get a third function. Composition is our most basic operation with functions. For instance, if

$$f(x) = 3x^2 \qquad \text{and} \qquad g(x) = 5x + 1$$

then we can plug $g(x)$ into $f(x)$ and create a third function, $h(x)$:

$$f(\,g(x)\,) = 3(\,g(x)\,)^2 = 3(5x+1)^2 = h(x)$$

This is what we do when we talk about linear transformations—we take a parent function, $f(x)$, and into it we either plug a different function (to make a horizontal transformation), or we plug the parent function into a different function (to make a vertical transformation). For instance, say we have the parent function $f(x)$ (without knowing exactly what it is), and say we have the transformation-function $g(x) = x - 3$. Then

$$f(\,g(x)\,) = f(\,g(x)\,) = f(x-3) = \text{the parent function shifted right by three}$$

The 'x' in a function sometimes stands for a position along the horizontal axis of a coordinate plane, but more generally, 'x' is just a "placeholder"[1] that can stand for anything. For instance, these four functions are all written with different placeholders, but they all describe the same underlying function, f:

$$f(x) = \frac{1}{x^2} \qquad\qquad f(t) = \frac{1}{t^2}$$

$$f(\text{Mr. Dickerson}) = \frac{1}{(\text{Mr. Dickerson})^2} \qquad\qquad f(\zeta) = 1/\zeta^2$$

So instead of plugging x or Mr. Dickerson into f, why not replace the placeholder with an entire function? This is the essence of composition.

Anyway, here's our real question: what if we want to *undo* composition of functions? Think about linear transformations: if we move something right by three, we can undo that action by moving it left by three. This question is hardly restricted to functions alone. We *do things* all the time, and so it is completely natural to ask, can we *undo things* (and if so, how)? For instance: if we add two things, we can undo that by subtracting, thus getting the first thing back:

$$A + B - B = A$$

If we multiply two things, we can undo that by dividing:

$$A \cdot B \cdot \frac{1}{B} = A$$

We usually don't use the word "undo" to describe this operation; we usually refer to it as *inverting* or *finding an inverse*. So here's our question: we know how to compose functions, but how do we invert that operation? how do we find an inverse for function composition? can we?

[1] I feel that this word is overused and underdefined, but I don't know any better words.

Let's do a quick example. Imagine I take something and quint it (i.e., raise it to the fifth power). How do I get the original thing back? I just take a quintic (fifth) root.

$$(x^5)^{1/5} = \sqrt[5]{x^5} = x$$

So quinting (raising something to the fifth) and taking a quintic root (taking a fifth root) are *inverse functions*. We know that because if we do one after the other, we get back the original thing we plugged into the equation. In this case that was just x, but it could well have been anything. Imagine that Mr. Dickerson, on a field trip to the quinting factory, accidentally fell into the quinting machine. In order to get Mr. Dickerson back, you'd just need to bring him to the quintic-root factory next door:

$$\sqrt[5]{(\text{Mr. Dickerson})^5} = \text{Mr. Dickerson}$$

Of course, it doesn't matter what order we compose these functions in[2]. If A is the inverse of B, then B is the inverse of A:

$$\sqrt[5]{x^5} = x$$
$$(\sqrt[5]{x})^5 = x$$

Formally, then, we define an **inverse function** in the following way: $f(x)$ and $g(x)$ are inverses if (and only if) the following are true:

- $f(\,g(x)\,) = x$, or
- $g(\,f(x)\,) = x$

Note that if one of these identities is true, the other will be true, too.

Often we use a special notation for inverses: if $f(x)$ is some function, then we write its inverse function as $f^{-1}(x)$. But this is bad notation. The superscript (-1) here has absolutely, absolutely nothing to do with a reciprocal. $\frac{1}{f(x)}$ is **not** (with one or two uninteresting exceptions) the inverse of $f(x)$. Yes, we do usually write reciprocals with a (-1) superscript. But here the (-1) just denotes an inverse. That's why it's horrible notation—the same symbol for two entirely different things. "In the language of everyday life," Wittgenstein writes,

> it very often happens that the same word signifies in two different ways—and therefore belongs to two different symbols—or that two words, which signify in different ways, are apparently applied in the same way in the proposition... Thus there easily arise the most fundamental confusions. In order to avoid those errors, we must employ a symbolism which excludes them, by not applying the same sign in different symbols, and by not applying signs in the same way which signify in different ways.[3]

(This quotation isn't necessary to prove my point; I'm only including it because it *is* apt, and I was reading it earlier today.[4]) Quite frankly, I'd rather you never use that notation at all; the only reason I'm mentioning it is because other people use it, so if you see an $f^{-1}(x)$ somewhere else in life, you'll know what they mean. If you need a good way of writing the inverse of $f(x)$, why not write $f^{\text{inv}}(x)$ instead? (Or come up with some other clever notation.)

Beyond defining what an inverse *is*, though, the useful question is: how do we actually *compute* them? There are two ways: algebraically and graphically[5].

[2]Not for our purposes, anyway—not for the types of functions we deal with, for the most part. I could give more complicated examples where g is the inverse of f but f isn't precisely the inverse of g.

[3]*Tractatus* 3.323-3.325

[4]Also, Wittgenstein uses the word "sign" to mean the same thing that I called a "symbol," and by "symbol" W. means something else altogether... so I am afraid I am guilty of the very offense both he and I believe should be capital.

[5]"Graphically" isn't really a method of "computation", but I think you know what I mean.

- **Algebraically**: we transpose (switch) x and y in the equation (and then solve for y). For example: if we have

  $$y = 2x^3 + 5$$

  then its inverse is

  $$x = 2y^3 + 5$$

  Usually, though, we write functions with the y isolated (i.e., solved for y in terms of x), and if we were to rearrange this, we'd get

  $$\begin{aligned} x - 5 &= 2y^3 \\ \frac{x-5}{2} &= y^3 \\ y &= \sqrt[3]{\frac{x-5}{2}} \end{aligned}$$

  So then the following are inverse functions

  $$y = 2x^3 + 5 \qquad \text{and} \qquad y = \sqrt[3]{\frac{x-5}{2}}$$

  I guess another way to denote this would be to say:

  $$f(x) = 2x^3 + 5 \qquad \text{and} \qquad f^{\text{inv}}(x) = \sqrt[3]{\frac{x-5}{2}}$$

  or

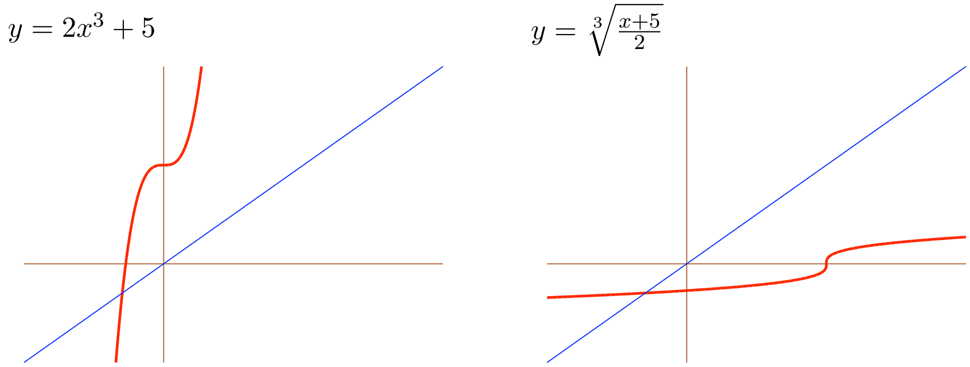
  $$f(x) = \sqrt[3]{\frac{x-5}{2}} \qquad \text{and} \qquad f^{\text{inv}}(x) = 2x^3 + 5$$

  If we wanted to formally *prove* that these are inverses, we'd need to show that they satisfy our definition of an inverse. So we'd need to plug them into one another, simplify, and end up just with x:

  $$\begin{aligned} f^{\text{inv}}(\,f(x)\,) &= \sqrt[3]{\frac{f(x)-5}{2}} \\ &= \sqrt[3]{\frac{(2x^3+5)-5}{2}} \\ &= \sqrt[3]{\frac{2x^3}{2}} \\ &= \sqrt[3]{x^3} \\ &= x \end{aligned}$$

  Ta-da!

- **Graphically**: finding an inverse just corresponds to a simple swap of the x and y axis (a 90-degree rotation left, followed by a horizontal reflection), which is all just the same as reflecting the function across the line $y = x$, like so:



Note that not every function has an inverse that is itself a function. Why not? Come up with an example of a function whose inverse isn't a function. For functions we have a "vertical line test" for functionality—can you come up with a similar test to automatically determine whether a given function has an inverse function? (This, by the way, is where commutativity of inverses begins to break down—where A being the inverse of B doesn't mean that B is the inverse of A.)

Here's another fun question: can you come up with a function(s) that is (are) its own inverse(s)?[6] what are the necessary algebraic/graphical properties of such a function?

Problems

Graph each of the following functions and try to sketch their inverses; then find the inverse algebraically. (If the inverse is a function, say so; if not, say why.) Then prove that the inverse is, in fact, an inverse.

1. $f(x) = -x$

2. $f(x) = -x + 1$

3. $f(x) = 5x - 12$

4. $f(x) = ax + b$

5. $f(x) = 5x^2 - 4$

6. $f(x) = 5 - 2x^3$

7. $f(x) = (x^5 + 1)^3$

8. $f(x) = \sqrt{4x - 7}$

9. $f(x) = 5 + \sqrt{3x - 2}$

10. $f(x) = 1/x$

11. $f(x) = 1/x^2$

12. $f(x) = x^k$

13. $f(x) = \dfrac{1}{2x^2 + 1}$

14. $f(x) = \dfrac{x}{x^2 + 1}$

15. $f(x) = \sqrt[5]{\dfrac{3x - 1}{x - 2}}$

16. $f(x) = ax^2$

17. $f(x) = a(x + b)^3 + c$

18. $f(x) = e^x$

[6] The mathematical name for such a function is an "involution."