Logarithms! And Exponentials.

Calculus 11, Veritas Prep.

Arizona. August 2010. It's 145 degrees. Wave after wave of photons crashes down on you as you stroll along the shore of Tempe Town Flats. It used to be a lake, but that was before—before it all evaporated. The water, the town, the people, their souls, your bank account—gone. All gone. They evaporated into a humid human mist that hung in the air briefly, like a firework at the apex of its flight, and then dispersed, constrained by the second law of thermodynamics to forever increase the entropy of the universe.

It's so hot you can't even ride your bike. The rubber tires crumble the instant they meet UV. Forget about driving. Car tires are just as bad. Prices have reached \$10,000 for a set, and even then, you can only drive between 1 and 2 AM. Your buddy stole a tank from the abandoned army base—no rubber on that thing!—but you wouldn't want to be sealed into a hermetic steel chamber in this heat. You no longer need superheated gases to weld your metal sculptures together. You just use a magnifying glass.

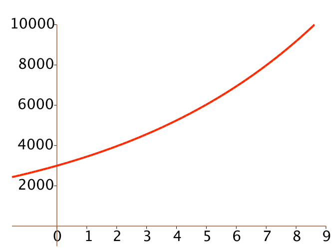
You've got to get out of Tempe. Luckily, you've saved up $\$3,000$ over the past few years by running a textbook-smuggling ring. New editions of glossy books, delivered in crates by your contacts at print shops in Dallas and Sacramento, resold at a tenth of retail to impoverished ASU students. Somehow the university still operates, despite what's happened to the rest of the Valley. Quality doesn't seem to have gone down.

You've got to get out of the wasteland. You want to go to Alaska. You're not really sure what you would do. Maybe set up a used-book store outside of Fairbanks. Maybe set up a homestead; live off the land. Lumber's probably easily available. You could export moose- and bear-jerky to gourmet food stores the Lower 48.

In any case, you figure you need $\$9,000$ to get up there and get set up. You've got a guy in Tucson who says he can given you a 15% return per year on your money. 15% per year, every year. That beats inflation. That beats inflation a lot. You decide to invest with him. Eventually, you'll have enough to export yourself to the land of the midnight sun.

time passed	your assets		
initially	$\$3,000$		
after 1 year	$\$3,000 \cdot (1.15)$	$= \$3,450$	
after 2 years	$\underbrace{\$3,000 \cdot (1.15)}_{\text{what you had last year}} \cdot (1.15)$	$= (\$3,000) \cdot (1.15)^2$	$= \$3,967.50$
after 3 years	$\underbrace{\$3,000 \cdot (1.15)(1.15)}_{\text{what you had last year}} \cdot (1.15)$	$= (\$3,000) \cdot (1.15)^3$	$= \$4,562.63$
after 4 years	$\underbrace{\$3,000 \cdot (1.15)(1.15)(1.15)}_{\text{what you had last year}} \cdot (1.15)$	$= (\$3,000) \cdot (1.15)^4$	$= \$5,247.02$
$\vdots$	$\vdots$		
after t years	$(\$3,000) \cdot (1.15)^t$		

After you invest, your assets will increase exponentially! You can even plot their growth with time:



Here's the question: when will you be able to go to Alaska? When will you have the $\$9,000$ necessary to leave? From the graph, it looks like it'll take another 8 years of textbook-running.

But how do you find this *exactly*? You need to find what value of t makes your asset-function equal to $\$9,000$:

$$\$9,000 = \$(3000) \cdot (1.15)^t$$

How do you solve this equation? I guess you could start by dividing off the $\$3,000$:

$$\frac{\$9,000}{\$3,000} = (1.15)^t$$

$$3 = (1.15)^t$$

But how do we isolate t? Could we take a t'th root?

$$3^{1/t} = ((1.15)^t)^{1/t}$$

$$3^{1/t} = 1.15$$

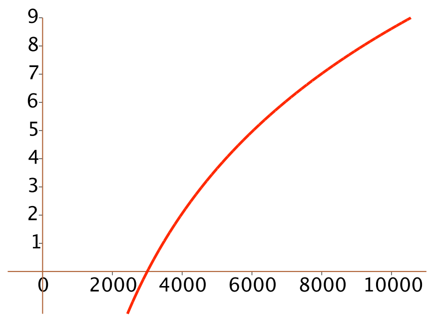
That doesn't work. It just gives us the same problem, in a different place. We still have a t in the exponent. What we really need, I guess, is some function that could get the t out of the exponent. Some function that could undo the action of exponentiation... *an inverse function*!

How would we do this? I guess we'd say that we have the function $f(x) = 3000 \cdot 1.15^x$, and we want to find $f^{\text{inv}}(x)$, such that $f^{\text{inv}}(3000 \cdot 1.15^t) = t$. That way, we could solve our equation for t:

$$9000 = 3000(1.15)^t$$

$$f^{\text{inv}}(9000) = f^{\text{inv}}(3000 \cdot 1.15^t)$$
$$f^{\text{inv}}(9000) = t$$
$$t = f^{\text{inv}}(9000)$$

Whatever that inverse function is—and we don't really know what it is; we're just assuming that exists and that it's *something*—we can just plug $\$9,000$ into it, and find out how long we need to wait until we triple our assets and can go to Alaska!!! I guess the inverse function will look like this (a reflection of the original function across $y = x$):



The problem is even more general than simply finding out when we can go to Alaska. More broadly, we know how to deal with exponential functions. Can we come up with functions that are the inverse of exponential functions? Let us call such functions (whatever they are) **logarithms**, and define them formally in this way, as the inverse of an exponential function: if we have an **exponential function with base a**

$$f(x) = a^x$$

then the **logarithm base a** is its inverse:

$$f^{\text{inv}}(x) = \log_a x$$

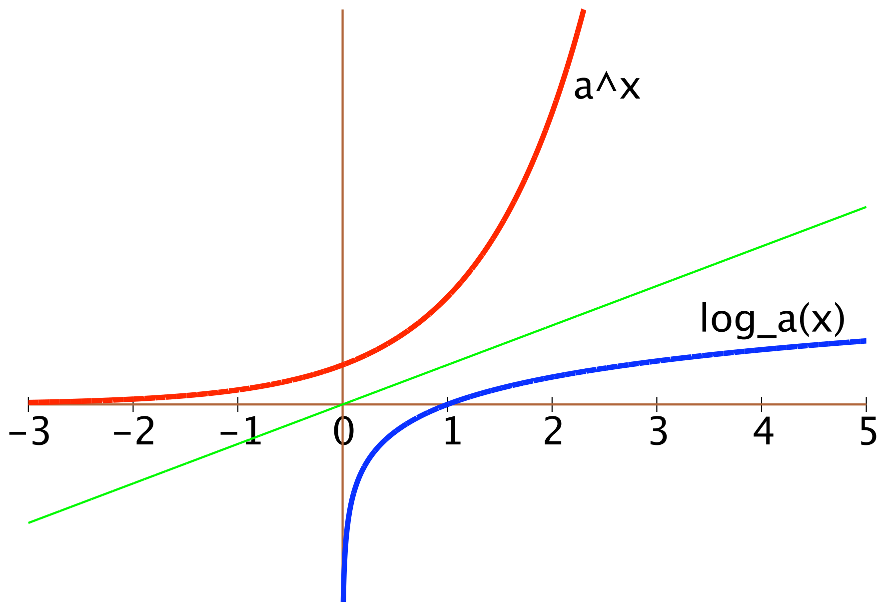
such that

$$\log_a(a^x) = x \qquad \text{and/or} \qquad a^{\log_a(x)} = x$$

We might not have a general method to figure these out—to actually *compute* logarithms as specific functions or specific numbers, written in terms of decimals and square roots and arithmetic operations and whatnot. But if we define the logarithm in this way, we should be able to figure out some simple ones. I'll give examples in a moment.

The problem is, if we define the logarithm in this way, we don't know very much about it. We don't know what the function is in terms of numbers and addition/subtraction/multiplication/division, in terms of x's and square roots and that sort of thing. All we know is that it's the inverse of an exponential. That, by itself, isn't particularly useful. We can't call up the travel agent and book tickets to Alaska, "departing $\log_{1.15}(9000)$ years from now." Not yet, at least. So the question is: if this is all we know about the logarithm, what else must necessarily be true?

I guess we must know what it looks like. We know what exponential functions look like (e.g. e^x, 5^x, etc.), and we know what inverses look like (they look like the original function flipped around the line $y = x$). So we can graph logarithms! Imagine we have some generic exponential function a^x, where a is some constant. Then the $log_a(x)$ must look like this (graphed together with $y = a^x$ and $y = x$):



So that's helpful. The graph should give us *some* idea as to how log should behave as a function—what sort of numbers it should give us when we plug numbers into it. There's a vertical asymptote at $x = 1$, and an x-intercept at $x = 0$. It looks like it increases as we go to the right, but slowly[1].

What else can we figure out about logarithms, based solely on this definition? I suppose the next most important observation (which concerns specific numbers and not more general functions) is this. Imagine we have some sort of exponential equation, like:

$$a^b = c$$

This is the same[2] as writing:

$$\log_a(c) = b$$

[1]Note that it doesn't exist on the left side of the graph. For our purposes, we're only defining the logarithm of positive numbers. You *can* define the logarithm for negative numbers, but for various reasons there are lots of different ways to define it, and it gets kind of messy. (Complex numbers get involved. The typical way to define the logarithm of a negative is such that $\ln(-1) = i\pi$.)

[2]By "the same" I mean that these two statements are logically equivalent—that if one is true, then the other is necessarily true as well. Often we just denote this using a double-double-arrow: $a^b = c \iff \log_a(c) = b$

For instance, since $5^2 = 25$, then $\log_5(25) = 2$. Why is this true (in general)? Well, imagine I have $a^b = c$. Then...

$$\begin{aligned}
a^b &= c \\
\log_a(a^b) &= \log_a(c) && \text{(taking the } \log_a \text{ of both sides)} \\
b &= \log_a(c) && \text{(they're inverse fxns—they just cancel out} \\
& && \text{on the left)} \\
\log_a(c) &= b && \text{(rearranging)}
\end{aligned}$$

So, for example:

- $5^2 = 25$, and $\log_5(25) = 2$
- $4^3 = 64$, and $\log_4(64) = 3$
- $2^4 = 16$, and $\log_2(16) = 4$

Another way to think about these examples is this:

- $$\begin{aligned} \log_5(25) &= \log_5(5^2) && \text{(another way to write 25)} \\ &= 2 && \text{(inverse fxns—they cancel out)} \end{aligned}$$
- $$\begin{aligned} \log_4(64) &= \log_4(4^3) && \text{(another way to write 64)} \\ &= 3 && \text{(inverse fxns cancel)} \end{aligned}$$
- $$\begin{aligned} \log_2(16) &= \log_2(2^4) && \text{(another way to write 8)} \\ &= 4 && \text{(inverse fxns cancel)} \end{aligned}$$

Yet another way to think about these (and I think this is the best way) is as a question:

- $\log_5(25)$ is like asking, $5^{\text{what}} = 25$? Clearly, the answer is just 2. So $\log_5(25) = 2$.
- $\log_4(64)$ is like asking, $4^{\text{what}} = 64$? Clearly, the answer is just 3. So $\log_4(64) = 3$.
- $\log_2(16)$ is like asking, $2^{\text{what}} = 16$? Clearly, the answer is just 4. So $\log_2(16) = 2$.

Here's another way of thinking about logs. (I'm worried this is really disorganized, so if you have better ideas, please let me know.) I like the number seven. It's my favorite number. But sometimes I don't want to write just "7". Sometimes I want to write it differently. Luckily, there are lots of different ways to write 7:

- 7
- $10 - 3$
- $\sqrt{49}$
- $28/4$
- $7 \cdot 1$
- the solution to the equation $x - 7 = 0$
- $343^{1/3}$

But two years before 7 became my favorite number, 5 was my favorite number. What if I wanted to combine these two? What if I wanted to write my current favorite number (7) in terms of my older favorite number (5)? I guess I could write:

- $5 + 2 = 7$
- $5 \cdot 1.4 = 7$

But what if I wanted to do this not using additon (like in the first example) or multiplication (like in the second), but by using my *third* binary operation—exponentiation? Put differently, what if I want to write:

$$5^{\text{something}} = 7$$

How do I figure out what that something in the exponent is? Put more formally, if I have $5^x = 7$, how do I solve for x? The cool obsevation here, that might have escaped your notice, is that **we can write any (positive) number, simply as any other positive number raised to some power**!!! And logarithms tell us how we can do that. It's the same as how we can write any number as some other number plus something:

$$\text{there is some number } x \text{ such that } 5 + x = 7$$

$$\text{we can find it using subtraction: } x = 7 - 5$$

or how I can write any number as some other number times something:

$$\text{there is some number } x \text{ such that } 5 \cdot x = 7$$

$$\text{we can find it using division: } x = 7/5$$

so we define the **logarithm** as the operation that does this—that is the inverse of exponentiation.

$$\text{there is some number } x \text{ such that } 5^x = 7$$

$$\text{we can find it using a logarithm: } x = \log_5(7)$$

Does any of this make any sense? I have given, like, five different intuitive explanations of a logarithm, plus a formal definition, and some wacky story about textbooks and Tucson. I'm trying to make it un-confusing, but I worry that excessive information will have the opposite effect.

operation	inverse operation
addition	subtraction
multiplication	division
exponentiation	logarithmancy

Anyway, your persisting question should be, "Well, it's all well and good to say that $\log_5(7)$ is the number such that $5^{\text{that number}} = 7$, but I still don't know what it is! We've just written "log"! We don't know what it is as a decimal!!!"

That is a good objection. I hope you agree, as we saw before, that we can find the logarithm as a decimal under the right circumstances: $\log_5(25) = 2$, because $5^2 = 25$. But this leaves us with a lot to be desired. What is $\log_5(7)$ as a decimal, anyway? And when can we can go to Alaska? From looking at the graphs back on pages 2 or 3, we can see that it'll take maybe 7 or 8 years. But we'd like to know the exact date, so we can buy plane tickets! How do we figure that out?

The analogy here is to division. Sometimes we can divide numbers and get really clean results: $30/6 = 5$. But sometimes we can't, and we have to approximate using long division: $30/7 \approx 4.285712\ldots$ It's the same with logarithms. So if we're faced with something like $\log_5(7)$, we can do two things:

- We might be able to approximate in our heads. For example, if we want to find $\log_{10}(1004)$, we know that's probably a little bit more than 3, because $10^3 = 1000$. Or if we want to find $\log_6(34)$, that's probably a little less than 2, because $6^2 = 36$.

- You could use your... your... calculator. Eek. Except your calculator only has **ln** and **Log** buttons, for the base-e and base-10 logs. What if you want to find $\log_5(7)$? As it turns out, there's a *change-of-base formula* that lets you write any logarithm in terms of any other logarithm. We'll prove it in a bit. But I'll give it to you now:

  $$\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$$

  So in this case, you could have

  $$\log_5(7) = \frac{\log_e(7)}{\log_e(5)} = \frac{\ln(7)}{\ln(5)}$$

  And you could type that into your stupid hunk of silicon. You get that $\log_5(7) \approx 1.209\ldots$

- Or you could use the change-of-base formula, coupled with this fun fact from higher calculus:

$$\ln(1+x) \;=\; \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n} x^n \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

  This, by the way, is basically how your calculator computes logs—it uses a series approximation like this. But I don't think you'd really want to do this by hand, because it'd take a long time, and it'd be boring...

I guess the broader comment I would make here is, who cares if we can't write it as a decimal? A decimal is (usually) just an approximation of a number, anyway. We'd far prefer to write 30/7 as 30/7 and not as $4.2857\ldots$. The former is the *actual number*; the latter is just a dirty photocopy. This sort of thing comes up all the time. There are very few numbers that we can write exactly as a decimal (like 2 or 5.68). There are far more numbers that we can't express exactly in decimal form—like $\sqrt{2}$ or π or even something basic like 1/3.

the actual number	its decimal approximation
46	46
π	$3.14159\ldots$
$\sqrt{2}$	$1.41421\ldots$
50/17	$2.94117\ldots$
5/4	1.25
$\sin(\pi/5)$	$0.58778\ldots$
$\log_5(7)$	$1.20906\ldots$
$\ln(5)$	$1.60943\ldots$

Before we go much further, I should probably comment on a special logarithm, and the special number that is its base. (Because I just used it in my examples, among other things.) That is: the **natural logarithm** $\ln(x)$, defined as the logarithm to the base e: $\ln(x) = \log_e(x)$, where e is an irrational, transcendental number (like π): $e \approx 2.718$... Incidentally, there's a cool mnemonic to remember the first 15 digits:

$$e \approx 2.7\ 1828\ 1828\ 45\ 90\ 45...$$

So it's just 2.7, followed by the year Andrew Jackson was elected president, then that again, followed by a $45-45-90$ right triangle (but rearranged).

The abbreviation ln, by the way, comes from the Latin *logarithmus naturalis*. You can pronounce it "ell-enn" or "linn" or even just "log," if you're of a certain bent. Often mathematicians will write "log" when they mean "log base e". But the button **Log(x)** on your calculator computes the base-10 log ($\log_{10}(x)$), and a lot of K-12 teachers say that "log" means "log base ten." I think this is silly. 10 is a horrifically *un*natural base for a logarithm. The only reason we use base 10 is because we count in units of ten, and the only reason we count in units of ten (base 10) is because we have ten fingers. What if you were a tapir? with only four toes on each of your front feet? Would you then use a base-8 number system? What if you wanted to count with your rear toes? Tapirs only have three toes on each rear foot. Would you then use a base-6 number system? Thus, you can assume that if I just say "log" without any qualification as to the base, I'm talking about the natural log. If I say "the artificial, evolutionarily-induced log," then I'm probably talking about the base-10 log. (Or going on a bizarre rant about faux-wood trim.)

But where does e come from? Why do we care about the natural log? Is it just one of those bizarre math-things that we're not supposed to be able to understand but have to accept? No. e is a number that eerily arises out of mathematical nature, like π. π is just the ratio of the circumference of a circle to its diameter. But it's not a nice, clean integer, or even a rational number. It's an infinitely-long, non-repeating, patternless decimal. Out of very basic geometry we get a very strange number[3].

e's origin is slightly more abstract. It comes from this question: is there a function that is its own slope? As it turns out—you won't know why for a bit—as it turns out, the slope of exponential functions like 5^x or 7^x is equal to the same exponential function back again, but times some constant:

$$\text{slope of } 5^x = (\text{some constant}) \cdot 5^x$$

And e is the number such that that constant is 1:

$$\text{slope of } e^x = 1 \cdot e^x = e^x$$

THAT is why we care about the number e. It's the number whose exponential function *is its own slope*!!! There's another amazing fact we'll prove in 12th grade:

$$e^{i\pi} + 1 = 0$$

This is known as **Euler's Identity**, after Leonhard Euler (1707–1783). It is wondrous and beautiful. It gives, in a single equation, a relationship between the five most important numbers in math—e, i, π, 0, and 1—the three most important operations—addition, multiplication, exponentiation—and the most important relationship—equality.

Anyway, let's talk more about how logs behave. We've talked a lot about how we can compute logarithms of specific numbers. But let's consider how logarithms behave in a more general sense. Here are a bunch of cool properties that we'll prove (and that you can use):

Our Logarithm Theorems/Properties

Theorem A: $\log_a(a^x) = x$

Theorem B: $a^{\log_a(x)} = x$

Theorem C: $\log_a(xy) = \log_a(x) + \log_a(y)$

Theorem D: $\log_a(x^k) = k \log_a(x)$

Theorem E: $\log_a\left(\dfrac{x}{y}\right) = \log_a(x) - \log_a(y)$

Theorem F: $\log_a(1) = 0$

Theorem G: $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$

Theorems A & B: $\log_a(a^x) = x$ and $a^{\log_a(x)} = x$
Proof: We already proved these, right after we gave the definition of the logarithm (as an inverse function). But let's demonstrate the proof again.

Imagine we have some exponential function, $f(x) = a^x$. Then, by definition, the inverse of $f(x)$ is the logarithm base a ("log base a" is just another name for "the inverse of a^x"). So:

$$f(x) = a^x \qquad f^{\text{inv}}(x) = \log_a(x)$$

But we know that, if we have any two functions that are inverses of each other, they will cancel out upon composition:

$$f^{\text{inv}}(\,f(x)\,) = x \qquad \text{and} \qquad f(\,f^{\text{inv}}(x)\,) = x$$

So then, for these specific two inverses, we will have:

$$\log_a(\,a^x\,) = x \qquad \text{and} \qquad a^{\log_a(x)} = x$$

[3]Incidentally: what's the length of the hypotenuse of a right triange whose other sides are 1?

Theorem C: $\log_a(xy) = \log_a(x) + \log_a(y)$

This is one of the coolest properties of logs—that we can split them up along multiplication! Or, put differently, that logarithms can *turn multiplication into addition.* In fact, we could (if we wanted to) define logarithms only as a function that has this property (rather than define them as the inverse of an exponential). That's how they were created—18th century mathematicians wanted an easier way to multiply number together, and tried to find a function that could turn a (potentially difficult) multiplication problem into an (easier) addition problem.

Proof: The proof is kind of cool. First, we'll use logs to find a different way to write x and y, and then multiply them together to find a different way to write xy. And then we'll hit that up with a log *again*, simplify, and poof! we'll get our answer.

I know (from Thm B) that I can rewrite x as :	x	$= a^{\log_a(x)}$
Likewise, I can rewrite y as:	y	$= a^{\log_a(y)}$
If I put them together, xy must be equal to this:	xy	$= a^{\log_a(x)}a^{\log_a(y)}$
but by properties of exponents, this must be just:	xy	$= a^{\log_a(x)+\log_a(y)}$
and if I take $\log_a$ of both sides:	$\log_a(xy)$	$= \log_a\left(a^{\log_a(x)+\log_a(y)}\right)$
on right side, the $\log_a$ and a cancel, b/c inverses:	$\log_a(xy)$	$= \log_a(x) + \log_a(y)$

Theorem D: $\log_a(x^k) = k\log_a(x)$

This theorem is really useful and really cool. It tells us that if we want to get something out of an exponent (which is our whole reason for coming up with logs in the first place), we can take *any* logarithm—we don't need to get the base right. This is a very, very useful property. It tells us, for instance, that $\ln(7^5) = 5\ln(7)$.

Proof:

$$\begin{aligned}
\log_a(x^k) &= \log_a(\underbrace{x \cdot x \cdots x}_{\text{k times}}) && \text{(by definition of exponentiation)}\\
&= \underbrace{\log_a(x) + \log_a(x) + \cdots + \log_a(x)}_{\text{k times}} && \text{(by Thm C—I can split logs up along multiplication)}\\
&= k \cdot \log_a(x) && \text{(multiplication)}\\
&= k\log_a(x)
\end{aligned}$$

Theorem E: $\log_a\left(\dfrac{x}{y}\right) = \log_a(x) - \log_a(y)$

This proof is a slightly different version of Theorem C. In Theorem C, we showed that logarithms turn multiplication into addition; here, we show that logarithms turn division into subtraction. (Which should make sense.) Plus, I think the proof is cool, because rather than going all the way back to first principles, we can just use Theorem C (in combination with Theorem D) to prove it.

Proof:

$$\begin{aligned}
\log_a\left(\tfrac{x}{y}\right) &= \log_a(x \cdot y^{-1}) && \text{(properties of exponents)}\\
&= \log_a(x) + \log_a(y^{-1}) && \text{(by Thm C)}\\
&= \log_a(x) + (-1)\log_a(y) && \text{(by Thm D)}\\
&= \log_a(x) - \log_a(y)
\end{aligned}$$

Theorem F: $\log_a(1) = 0$
Proof: We know, as a property of exponents, that any number raised to the 0 is 1. Boom.

Theorem G: $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$

This is the change-of-base formula! With this formula, we'll finally be able to use our calculators (ew ew ew) and evaluate logarithms to weird bases—we'll be able to compute $\log_5(7)$, and we'll be able to compute when we can finally go to Alaska!

Proof: So imagine we want to rewrite $\log_a(x)$ using a differently-based logarithm (say in terms of a base-b logarithm). For starters, I hope you agree that, due to Theorem B, I can rewrite x as:

$$x = a^{\log_a(x)}$$

This may seem obvious and pointless, but be patient. Now imagine that we take the $\log_b$ of both sides. Then we have:

$$\log_b(x) = \log_b\left(a^{\log_a(x)}\right)$$

But because of Theorem D (I told you it was cool), we can just move that exponent inside the log out of it:

$$\log_b(x) = \log_a(x) \cdot \log_b(a)$$

But if we just rearrange this, we get:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

So now we can finally compute $\log_5(7)$, as well as figure out when we'll go to Alaska! For the former...

$$\log_5(7) = \frac{\ln(7)}{\ln(5)} \approx 1.20906\ldots$$

So then $5^{1.20906\ldots} = 7$!!! As for Alaska, we needed to solve the following equation for t:

$$\begin{aligned}
\$9,000 &= \$(3000) \cdot (1.15)^t \\
\frac{\$9,000}{\$3,000} &= 1.15^t \\
3 &= 1.15^t \\
\ln(3) &= \ln(1.15^t) \\
\ln(3) &= t\ln(1.15) \\
t &= \frac{\ln(3)}{\ln(1.15)} \\
t &\approx 7.8606\ldots
\end{aligned}$$

So we can go to Alaska in 7.86 years!!! Note that (secret of secrets) we didn't even need to use the change-of-base formula—we could have taken a $\log_{1.15}$, yes, and then used the change-of-base formula, but since we just wanted to get the t out of the exponent, *any* logarithm would do, so we chose the easiest one (the natural log).

Problems

Evaluate the following logarithms (without a calculator!):

1. $\log_2(2)$

2. $\log_5(5)$

3. $\log_2(2^3)$

4. $\log_7(7^9)$

5. $\log_{11}(11^{45})$

6. $\log_{15}(15)$

7. $\ln(e)$

8. $\ln(e^3)$

9. $\log_2(2^{10})$

10. $\log_4(4^3)$

11. $\log_5(25)$

12. $\log_3(81)$

13. $\log_7(343)$

14. $\log_2(512)$

15. $\log_4(64)$

16. $\log_{10}(10,000,000)$

17. $\log_{125}(1)$

18. $\log_4(1)$

19. $\log_7(1)$

20. $\log_{23,344.1}(1)$

21. $\log_\pi(1)$

22. $\ln(1)$

23. $\log_{246.7}(1)$

24. $\log_{27.9}(27.9)$

25. $\log_{46}(46)$

26. $\log_3(3)$

27. $\log_5\left(\frac{1}{5}\right)$

28. $\log_7\left(\frac{1}{7}\right)$

29. $\log_4\left(\frac{1}{4}\right)$

30. $\log_6\left(\frac{1}{36}\right)$

31. $\log_9\left(\frac{1}{81}\right)$

32. $\log_{11}\left(\frac{1}{121}\right)$

33. $\log_{12}\left(\frac{1}{144}\right)$

34. $\log_2\left(\frac{1}{4,096}\right)$

35. $\log_5\left(\frac{1}{125}\right)$

36. $\log_{1/2}\left(\frac{1}{2}\right)$

37. $\log_{1/6}\left(\frac{1}{6}\right)$

38. $\log_{1/7}\left(\frac{1}{7}\right)$

39. $\log_{1/9}(9)$

40. $\log_{1/5}(5)$

41. $\log_{1/3}(3)$

42. $\log_{1/5}(125)$

43. $\log_{1/9}(81)$

44. $\log_{1/4}(16)$

45. $\log_{1/9}(3)$

46. $\log_{1/16}(2)$

47. $\ln(\frac{1}{e})$

Estimate the following logarithms WITHOUT A CALCULATOR!!! Then use the change-of-base formula to estimate them with your calculator.

48. $\log_{10}(947)$

49. $\log_2(65)$

50. $\log_2(131)$

51. $\log_5(33.1)$

52. $\log_7(45)$

53. $\ln(3)$

54. $\ln(2.5e)$

55. $\ln(10)$

56. $\ln(10e)$

57. $\log_{10}(15)$

58. $\log_{10}(7500)$

59. $\log_3(3)$

Simplify the following logarithms/logarithmic expressions:

60. $\log_a(\frac{a^b}{a^c})$

61. $\log_a(a^b a^c)$

62. $\log_x(x^5)$

63. $\log_a(a^b + a^b)$

64. $\log_k(5^{2000})$

65. $\log_a(a + 4a^2)$

66. $\log_a(a^b k^b)$

67. $\log_a(a\sqrt[b]{a})$

68. $\ln(2k^5)$

69. $\log_a(\frac{xy^2}{4x^3})$

70. $\log_a(a^b b^5)$

71. $\log_a(\frac{a^b}{b^5})$

72. $\log_a(\frac{a}{50})$

73. $\log_a(10^5 5^3)$

74. $\ln(25^2 9^5)$

75. $\log_a(2^2 3^3 5^2 7^2 11^5)$

76. $\ln(5^{10} \cdot 3^{10} \cdot 2^{10} \cdot 7^{10} \cdot 100 \cdot 13^{10})$

77. $\log_a(x^2 - 2x - 3)$

78. $\ln(12x^3 + 9x^2 + 8x + 6)$

79. $\log_a(3x^2 + 5x + 11)$

80. $2\ln(\sqrt{e})$

81. $\ln(\ln(e^e))$

82. $\ln(e^{2\ln(x)})$

83. $\ln(\sin\theta) - \ln(\frac{\sin\theta}{5})$

84. $\ln(3x^2 - 9x) + \ln(\frac{1}{3x})$

85. $\frac{1}{2}\ln(4t^4) - \ln(2)$

86. $\ln(\frac{1}{\cos\theta}) + \ln(\cos\theta)$

87. $\ln(8x + 4) - 2\ln(2)$

88. $3\ln(\sqrt[3]{t^2 - 1}) - \ln(t + 1)$

89. $e^{\ln(7.2)}$

90. $e^{-\ln(x^2)}$

91. $e^{\ln(x)-\ln(y)}$

92. $e^{\ln(x^2+y^2)}$

93. $e^{-\ln(.3)}$

94. $e^{\ln(\pi x)-\ln(2)}$

Solve the following equations for y or t:

95. $\ln(y - 40) = 5t$

96. $\ln(y - 1) - \ln(2) = x + \ln(x)$

97. $\ln(y) = -t + 5$

98. $\ln(1 - 2y) = t$

99. $\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x)$

100. $e^{2y} = 4$

101. $100e^{10y} - 200 = 0$

102. $e^{5y} = 1/4$

103. $80e^y = 1$

104. $e^{y/1000} = a$

105. $e^{y\ln(.8)} = .8$

106. $e^{-.3t} = 27$

107. $e^{ky} = 1/2$

108. $e^{t\ln(.2)} = .4$

109. $e^{ky} = 1/10$

110. $e^{t\ln(2)} - 1/2 = 0$

111. $\ln(t) = 2t + 4$

Here are some word problems involving exponential growth and decay. Do them.

112. After doing some sort of bacterial experiment, you place your agar-coated petri dish underneath a UV lamp to sterilize it so that you can reuse it. You leave it under the UV lamp for ten minutes, and the high-energy light kills all the bacteria... except for one. The lone remaining bacterium, fed by the plentiful supply of agar, begins to grow and reproduce. It takes about an hour for it to undergo a complete cell cycle, so an hour later, there are two bacteria. How many bacteria are there after two hours? three hours? four hours? what about after t hours? what about after m minutes? how many bacteria will there be after a day? when will there be ten million bacteria on your petri dish?

113. Russia has a problem. A problem with overpopulation. Or rather, a problem with *under*population. Population growth in Russia is (and has been for the past several decades) below replacement levels, meaning that the population has actually been *decreasing*. (Somewhat hilariously, this is due not only to a declining birthrate, but also to a decreasing life expectancy, thanks to the increasingly third-world-quality of Russia's healthcare system (not to mention its political system).) In 2009, Russia's population was about 142 million, and was declining at a rate of 0.177% per year. If this is true, what should Russia's population be this year (2010)? next year? in 2011? what about in 2020? what about in the centenary of Putin's birth in 2052? when will the Russian population drop below 100 million? according to the model, when will it drop below 75 million? If you were a Russian policymaker, what would you do to alleviate this problem?

114. In 1985, the only freeways that existed in the greater Salt River Valley area[4] were I-10 and I-17, totaling about 85 miles in length. In 1985, Arizona voters passed Proposition 300, enacting a half-cent sales tax to fund highway construction. This resulted in the current car-encouraging sprawl of superhighways. In 2005, the total freeway network in Maricopa County was about 185 miles in length[5]. Assuming you have no other knowledge about Ph****x's growth patterns, Arizona politics,

[4] I refuse to call it the "Valley of the Sun," which emphasizes the superficial desires of local immigrants for "good weather" and ignores a perfectly competent, geographically-accurate, existing name.

[5] these data come from some Google Maps calculations, as well as AZDOT's 2005 Annual Report on Proposition 400, http://www.azdot.gov/Highways/valley_freeways/US60/Superstition/PDF/ANNUALREPORT89292.pdf

American demographics, etc., and can only assume that the Maricopa highway network will continue to grow at the same rate, how many miles of freeway should there be this year (2010)? What about in 2015? What about in 2020? What about in year t? When will the highway network in the Salt River Valley reach 1,000 miles in length?

115. After graduating from high school, you gradually begin to lose touch with friends, such that every year, you only have half as many friends from high school as you did the year before. At the instant of your graduation, you had 22 friends from high school. How many do you have the summer after your first year of college? the summer after that? Come up with a function for the number of high school friends you have as a function of the years since your graduation. (Do you think that creating said functions might be responsible for the decline in your number of friends?) When will you only have four friends left from high school? When will you have no friends left from high school?

116. One of the (viscerally) coolest physics labs I did at the University of Chicago (cooler even than my experiment with polymers that required commandeering my dorm's industrial-size freezer) involved creating a radioactive isotope and measuring its half life. In the basement of the undergraduate physics building in a lead-lined room is a plutonium-beryllium "neutron howitzer". The howitzer's radioactive ingredients spit out a steady stream of neutrons. If you were to stand next to it for too long, your atoms would absorb some of the neutrons and become radioactive, too[6].

So my friend Alex and I went down to the basement with a small piece of silver, and stuck it into the neutron howitzer. Silver, as it turns out, is an excellent material for this sort of experiment. In its natural state, it is a mixture of two stable isotopes, Ag^{107} and Ag^{109}. Adding a neutron to each produces radioactive isotopes, Ag^{108} and Ag^{110}. Each of these isotopes will eventually decay by spitting the neutron back out.

Before we go any further, let's talk about half-life. Radioactive decay is fundamentally quantum-mechanical. If you have something like Ag^{108}... well, anyway, I just realized that this problem is going to be really hard, because these two isotopes have different half-lives, and so the tricky thing in the experiment was isolating from the data the decay from each of the isotopes, etc., and measuring the half-life of each of them... I guess it would be helpful if I had the actual lab report in front of me. But it's back in Ithaca. I'm just making this up as I remember it. Hm. Well, anyway, the point is that it was a fun way to end a year of general physics.

117. Between 2003 and 2007, the University of Chicago's endowment increased from \$3.22 billion dollars to \$6.2 billion dollars—an average increase of roughly 18% per year[7] . Assuming that the U of C's endowment increases exponentially, as financial instruments seem wont to do in first-order approximations, come up with a function for the value of the U of C's endowment as a function of time. (It might help to set 2003 as "year zero," though if you set the year 0 C.E. as "year zero" in the model, it's just a simple horizontal shift away.) Based on your model, what was the endowment worth in 2006? When did the endowment hit \$5 billion? What should the endowment have been worth in 2009, according to your model? The actual value of the endowment in 2009 was \$5.1 billion. Why does your model disagree?

118. You have a baby. You want the baby to go to college. (You hope the baby wants to go to college, too.) But college is expensive, so as soon as the baby is born, you start putting aside money into an educational investment account (a 529 plan[8]). You're not sure how much money to put in, so you do some calculations.

(a) Imagine you deposit k dollars into this account every year, and the account grows at a nice,

[6] Well, there are two options: either your atoms could absorb a neutron and be boosted into a heavier-but-still-stable isotope, or they could absorb a neutron and be boosted into a heavier-but-unstable isotope, which will eventually radioactively decay by spitting the extra neutron back out.

[7] 2009 University of Chicago Annual Report, http://www.uchicago.edu/annualreport/financials/endowment.shtml

[8] See section 529 of the Internal Revenue Code, 26 U.S.C. §529

predictable rate of p percent every year. (So after the first year, the account is worth $k(1+p)$.) How much money will there be in 18 years, when your kid is headed off to Cambridge? Give your answer in terms of k and p. If you assume that it grows at a reasonable rate of 5% every year[9], what will a \$1,000 annual contribution become in 18 years? A \$5,000 annual contribution? \$10,000? The current cost of attending MIT for four years is about \$200,000—how much money will you need to deposit every year in order to have that much in 18 years?

(b) Realistically, though, the cost of attending MIT will increase in the next 18 years. Assume that the total four-year cost of attending school X is currently c dollars, and that it increases at a steady rate of q percent each year. How much will it cost to attend school X in 18 years? Give your answer in terms of c and q. What will the cost of attending MIT be in 18 years?

(c) Let's combine these two ideas. Your kid wants to attend school X in 18 years, and you deposit k dollars annually into an account that grows at p percent per year. How much money should you deposit per year? Give your answer in terms of k, p, c, and q. Now imagine that school X is MIT, and that your kid's 529 plan returns 5% per year. How much should you deposit per year?

(d) Of course, realistically, your income is probably going to increase over the course of the 18 years this smelly little thing is wandering around your house, so we could continue this and say, well, what if our contribution to the 529 plan increases by r percent each year... but that would be creating a very complicated model all teetering on top of this assumption of nice, predictable, exponential growth. And economies and incomes don't actually work like that. Also, MIT gives out lots of financial aid.

[9] not historically unreasonable, but then again, "past performance is not a predictor of future returns."