Exponentiation: Theorems, Proofs, Problems

Pre/Calculus 11, Veritas Prep.

Our Exponentiation Theorems

Theorem A: $a^{n+m} = a^n a^m$	**Theorem E:** $\dfrac{a^n}{a^m} = a^{n-m}$
Theorem B: $(a^n)^m = a^{nm}$	**Theorem F:** $a^0 = 1$
Theorem C: $(ab)^n = a^n b^n$	**Theorem G:** $a^{-n} = \dfrac{1}{a^n}$
Theorem D: $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	**Theorem H:** $a^{n/m} = \sqrt[m]{a^n}$

Like all theorems, these do not come out of nowhere. They come from a definition and logical deduction.

Let us start, then, with a definition. What is *exponentiation*, anyway? What do we mean when we write a number with a superscript? I contend that we mean[1] something like *repeated multiplication*. When we write, for example, 5^4, we mean this simply as a more convenient way of writing $5 \cdot 5 \cdot 5 \cdot 5$ (or, even more simply, 625). Formally, let's define exponentiation in the following way[2]:

$$a^n \equiv \underbrace{a \cdot a \cdot a \cdots a}_{\text{n times}}$$

(The triple-equals-sign here is used to show that this is a definition—that this equation is not the consequence of some earlier theorem or axiom, but that with it we're *defining* what we mean by a^n.) Let us prove these theorems.

Theorem A: $a^{n+m} = a^n a^m$

Proof: We'll start with the left side of the equation, apply the definition of exponentiation, do some algebra, and eventually end up with the right side.

$a^{n+m} = \underbrace{a \cdot a \cdot a \cdots a}_{\text{n+m times}}$	(by the definition of exponentiation)
$= (\underbrace{a \cdot a \cdot a \cdots a}_{\text{n times}})(\underbrace{a \cdot a \cdot a \cdots a}_{\text{m times}})$	(because multiplication is associative)
$= a^n \cdot a^m$	(applying the definition of exponentiation again)
	(done! this stylized A is my end-of-proof symbol.)

Theorem B: $(a^n)^m = a^{nm}$

Proof: To prove this, we'll need to apply the definition of exponentiation—twice. (Well, actually three times, but the last time doesn't count.)

[1]For the most part. If you're a mathematician, we could have a lengthy conversation about everything I say on this sheet, but for our purposes this definition is sufficient. If you disagree (and this is in an imaginary conversation between myself and a mathematician), I will say but this: do you believe we should teach students Dedekind cuts before we let them utter the words, "real number"?

[2]Note that we haven't put any specifications on what a and n can and/or should be—must they be integers? real numbers? complex numbers? The way our definition works, what with the "n times" business, it must only hold for n being a positive integer, but these theorems hold for n being any real number. What we'll see in our derivations is that, even though we start by considering n only as a positive integer, we can extend our idea of "exponentiation" in a very logical way such that it holds for any real number. But this is in a footnote for a reason.

$$
\begin{aligned}
(a^n)^m &= (\underbrace{a \cdot a \cdot a \cdots a}_{\text{n times}})^m && \text{(by definition)}\\
&= \underbrace{(\underbrace{a \cdot a \cdot a \cdots a}_{\text{n times}}) \cdot (\underbrace{a \cdot a \cdot a \cdots a}_{\text{n times}}) \cdots (\underbrace{a \cdot a \cdot a \cdots a}_{\text{n times}})}_{\text{m times}} && \text{(by definition, again)}\\
&= \underbrace{a \cdot a \cdot a \cdots a}_{\text{n*m times}} && \text{(multiplication)}\\
&= a^{nm} && \text{(finish by re-applying the definition in reverse)}
\end{aligned}
$$

Theorem C: $(ab)^n = a^n b^n$
Proof: See a pattern? We'll apply the definition of exponentiation, do some algebra, and eventually get what we want.

$$
\begin{aligned}
(ab)^n &= \underbrace{ab \cdot ab \cdots ab}_{\text{n times}} && \text{(by definition)}\\
&= (\underbrace{a \cdot a \cdots a}_{\text{n times}}) \cdot (\underbrace{b \cdot b \cdots b}_{\text{n times}}) && \text{(multiplication is commutative —we can rearrange)}\\
&= a^n b^n && \text{(definition)}
\end{aligned}
$$

Theorem D: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
Proof: How do you think this should go?

Theorem E: $\frac{a^n}{a^m} = a^{n-m}$.
Proof: This is where things start to get interesting. The clear first step is to write out the entire quantity:

$$
\frac{a^n}{a^m} = \frac{\overbrace{a \cdot a \cdots a}^{\text{n times}}}{\underbrace{a \cdot a \cdots a}_{\text{m times}}}
$$

But where we go from here will depend on the relative size of n and m. n could be bigger than m, it could be smaller, or the two could be equal.

Let's start with the first case—that of $n > m$. In that case, both the top and the bottom will have at least m of the a's, and the top will have more— it'll have $n - m$ more (because $m + (n - m) = n$). Put differently, we can think of our fraction as being something like this:

$$= \frac{a^n}{a^m} = \frac{\overbrace{a \cdot a \cdots a}^{\text{m times}} \cdot \overbrace{a \cdot a \cdots a}^{\text{n-m times}}}{\underbrace{a \cdot a \cdots a}_{\text{m times}}}$$

If you're confused, count up the total number of a's on top—there should be n of them, just like what we started with. And then—wait a second! There are m a's on top, and m a's on the bottom! We can cancel them out! And we just end up with:

$$= \overbrace{a \cdot a \cdots a}^{\text{n-m times}}$$

Which, by definition, is just a^{n-m}.

But what about these other two cases? First of all, what happens in the case that $m = n$? Well, if $m = n$, then $\frac{a^n}{a^m}$ will just equal $\frac{a^n}{a^n}$ (or $\frac{a^m}{a^m}$—either way is the same). But that clearly[3] is just equal to 1. And I hope you will agree that if n and m are equal, then $n - m = 0$, and that then $a^{n-m} = a^0$. So clearly, then, since we want $\frac{a^n}{a^m} = a^{n-m}$, then $a^0 = 1$. (Which should make sense. 1, after all, is the multiplicative identity, so if we multiply something "no" times, we should end up not with zero, but with one. Think about how if you have a fraction in which everything cancels out, you have 1, not 0. Same sort of deal.)

Finally. What if $n < m$? Then (by the same argument as the first case) we have something like:

$$= \frac{a^n}{a^m} = \frac{\overbrace{a \cdot a \cdots a}^{\text{n times}}}{\underbrace{a \cdot a \cdots a}_{\text{n times}} \cdot \underbrace{a \cdot a \cdots a}_{\text{m-n times}}}$$

But wait! We can cancel things here, too! We get:

$$= \frac{1}{\underbrace{a \cdot a \cdots a}_{\text{m-n times}}} = \frac{1}{a^{m-n}}$$

But this isn't quite what we want. We want a^{n-m}. So let's define $\dfrac{1}{a^{\text{stuff}}}$ as being equal to $a^{\text{-stuff}}$. That way, our equation here will become:

$$= \frac{1}{a^{m-n}} = a^{-(m-n)} = a^{-m+n} = a^{n-m}$$

Excellent.

This has been a long discussion, so let's review what we've done. We've extended our idea of an exponent from just the positive integers to all integers (i.e., we can now exponentiate by 0 and negative numbers!) But in order to do this, we had to make a slight extension of our definition. Our original definition only considered the case of an exponent being a positive integer; in the course of this proof, we discovered a natural way to extend that definition to 0 and the negatives. Thus this has been somewhat more than just a "proof"—it has been partly a proof (for $n > m$), and partly a redefinition.

Theorem F: $a^0 = 1$
Proof: We already discussed this, in the proof of **E**

Theorem G: $a^{-n} = \dfrac{1}{a^n}$
Proof: Again, we already discussed this in the proof of **E**

[3]Note how my writing style is a bit different in this proof than in the previous ones—I'm making an argument in prose rather than in a nice, clean, two-column format. That's OK. Many math textbooks do the same thing, and it's the job of the reader (and what a job it is!) to translate that into a more clear, more symbolic format if necessary.

Theorem H: $a^{n/m} = \sqrt[m]{a^n}$

Proof: So far, we already know how to deal with exponents when the exponents are positive integers (thanks to our definition), zero (thanks to **F**), or negative integers (thanks to **G**). But what if we want to move beyond the integers and have exponents that are rational numbers (i.e., fractions)? This theorem will tell us how to think about that.

First of all, I should point out that we don't really have a solid definition of what we mean by $\sqrt{\text{stuff}}$. (From a technical, mathematical standpoint, what we're doing here is *defining* what we mean by such notation.) But intuitively, I hope you agree that the basic idea of a radical/root is that if I have an nth root of something, and I multiply n of those nth roots, I end up with my original thing. Meaning the thing under the radical sign. Right? Good.

So imagine we have $a^{1/m}$. (Can we do this? Let's assume we can.) Now imagine we have m copies of it, all multiplied together:

$$\underbrace{a^{1/m} \cdot a^{1/m} \cdot a^{1/m} \cdots a^{1/m}}_{\text{m times}}$$

By the definition of exponentiation, this must be $(a^{1/m})^m$. But by **B**, we can simplify this:

$$a^{1/m \cdot m} = a^{m/m} = a$$

In other words, whatever this $a^{1/m}$ is, if we take m copies of it, we get a. So this is just a root! (The mth root, to be specific). Formally:

$$a^{1/m} = \sqrt[m]{a}$$

What about the rest (the n/m part?) Imagine we have $a^{n/m}$. Because of Theorem **B** again, we can write this as $(a^n)^{1/m}$. Which, because of what we just proved, is $\sqrt[m]{a^n}$.

So now we know how to exponentiate by any rational number. Hooray!

Problems

Evaluate each of the following expressions.

1. 4^{-3}

2. 5^{-3}

3. $\left(\frac{2}{3}\right)^{-3}$

4. $9^{5/2}$

5. $9^{-3/2}$

6. $16^{-3/4}$

7. $64^{-2/3}$

8. $\left(5^0\right)^{2/3}$

9. $\left(7^{1/3}\right)^0$

10. $81^{-3/4}$

11. $\left(\frac{1}{27}\right)^{1/3}$

12. $\left(\frac{8}{27}\right)^{1/3}$

13. $\left(\frac{4}{25}\right)^{1/2}$

14. $\left(\frac{1}{125}\right)^{-2/3}$

15. $16^{3/4}$

16. $\left(121^0\right)^{1/2}$

17. $5^{-1} \cdot 5^{-3}$

18. $5^{-2} \cdot 6^{-2}$

19. $4^{-3} + 8^{-2}$

20. $6^0 + 6^{-1}$

21. $\left(3^0 + 3\right)^{1/2}$

22. $-9^0 + 9^{1/2}$

23. $\left(\sqrt[3]{216}\right)^2$

24. $81^{1/2} - 81^{-1/2}$

25. $\left(8^{243,458} + 121^{437}\right)^0$

26. $\left(15^0 + 15\right)^{1/2}$

27. $\left(3^{-1} + 3^{-2}\right)^{-1}$

28. $\frac{1}{4^{-2}} + \frac{1}{4^{-1}}$

29. $\frac{64}{64^{2/3}}$

30. $\frac{9^{1/2}}{27^{-1/3}}$

31. $5\sqrt{20} - \sqrt{45} + 2\sqrt{80}$

32. $\sqrt[3]{40} + 2\sqrt[3]{135} - 5\sqrt[3]{320}$

33. $\frac{2^{11/12} \cdot 2^{-7} \cdot 2^{-5}}{2^3 \cdot 2^{1/2} \cdot 2^{-10}}$

34. $\frac{(3^2)^{-1/2}(9^4)^{-1}}{27^{-3}}$

Simplify each of the following expressions so that they are written only with positive exponents and no radicals.

35. $\dfrac{5}{y^{-3}}$

36. $\dfrac{x^{-8}}{x^{-2}}$

37. $2x^{-1}y^{-2}y$

38. $(3x^{-2}y)(4x^5y^{-4})$

39. $(2x^3y)^{-2}$

40. $(3a^{-1}b)^{-3}$

41. $(16x^{-2})^{1/2}$

42. $(8y^6x^{-3})^{1/3}$

43. $(4x^{-1})(2x)^{-2}$

44. $\dfrac{x^2}{x^{-2}y^{-1}}$

45. $\left(\frac{2x}{3y}\right)^{-2}$

46. $\dfrac{18x^{-2}}{9xy^{-3}}$

47. $(27x^{-3}y^{-9})^{1/3}$

48. $\sqrt[4]{r^8s^{12}}$

49. $\sqrt[3]{27a^{-3}b}$

50. $\sqrt[7]{-x^{14}y^{28}}$

51. $\sqrt[9]{(4x+2y)^{18}}$

52. $\sqrt[3]{a+b} \;+\; \sqrt[3]{-(a+b)^2} \;+\; \sqrt[3]{a+b}$

53. $\dfrac{(7a)^2(5b)^{3/2}}{(5a)^{3/2}(7b)^4}$

54. $\dfrac{(6a)^{1/2}\sqrt{ab}}{a^2b^{3/2}}$

55. $\left(\dfrac{r^{2/3}}{s^{1/5}}\right)^{15/9}$

56. $(c^{2/5}d^{-2/3})(c^6d^3)^{4/3}$

57. $\dfrac{(2a)^{1/2}(3b)^{-2}(4a)^{3/5}}{(4a)^{-3/2}(3b)^2(2a)^{1/5}}$

58. $(a^{x^2})^{1/x}$

59. $\dfrac{(b^x)^{x-1}}{b^{-x}}$

60. $\dfrac{c}{(c^{5/6})^{42}(c^{51})^{-2/3}}$

Prove that each of the following equations are true. (How do you do this? Think of it as being like a two-column proof in geometry. Start with one side of the equation and step-by-step apply exponentiation laws until you end up with the other side of the equation.)

61. $\dfrac{\sqrt{ab}}{a^2b^{3/2}} = \dfrac{1}{ab\sqrt{a}}$

62. $\dfrac{\sqrt{c^2d^6}}{\sqrt{4c^3d^{-4}}} = \dfrac{d^5}{\sqrt{4c}}$

63. $\dfrac{a^ab^7}{(a\sqrt{b})^{14}} = \dfrac{1}{a^{14-a}}$