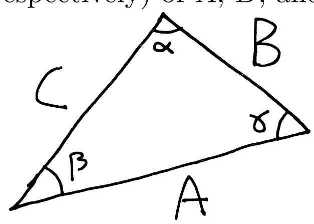
The Law of Sines

Pre/Calculus 11, Veritas Prep.

So we've learned a lot of trig so far. And it's been awesome. The useful thing about trig is its ability to turn problems of geometry into problems of algebra—and thus reduce all of our knowledge of the physical world to just equations with numbers[1]. But the major downside to the trig functions we've worked with so far is that *they only work with right triangles.* And yet many (most?) triangles aren't right triangles!!! You've seen plenty of word problems in which the major element of interest has been a triangle that didn't have a 90° angle. And you've dealt with these by turning them into right triangles—by drawing a few more sides and angles to make right triangles, then applying trig functions to them, and using that information to figure out whatever it was you wanted to figure out about your original, non-right triangle. I could probably add in a relevant word problem here to motivate this further.

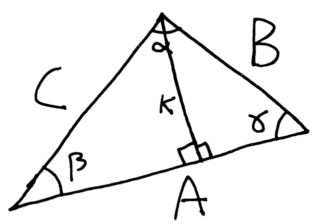
Anyway, we can codify that procedure—finding things out about a non-right-triangle by turning it into right triangles—in this way. Imagine we have some triangle—doesn't have to be a right triangle—with angles α, β, and γ, and opposite sides (respectively) of A, B, and C, like so:



Then the following equation(s) are true:

$$\frac{\sin(\alpha)}{A} = \frac{\sin(\beta)}{B} = \frac{\sin(\gamma)}{C}$$

Why? Imagine we have such a triangle (as seen above), and we turn it into two right triangles by drawing a line of length k, like so:



then, just from our knowledge of right-triangle trig, we must have

$$\sin(\beta) = \frac{k}{C} \qquad \text{and} \qquad \sin(\gamma) = \frac{k}{B}$$

If we rearrange both of these equations, we get:

$$k = C\sin(\beta) \qquad \text{and} \qquad k = B\sin(\gamma)$$

so I can set them equal to each other:

$$C\sin(\beta) = B\sin(\gamma)$$

and divide:

$$\frac{\sin(\beta)}{B} = \frac{\sin(\gamma)}{C}$$

Ta-da! To prove the rest of this—to show that these are both equal to $\frac{\sin(\alpha)}{A}$—we can do the same procedure, but this time draw a line that doesn't chop into the angle α. (For example, we could draw a line from the point near angle β, and get two right triangles, one with an angle α and another with an angle β.)

[1]Okay, this is way overblown, but I think you get the idea