Exactly Evaluating Even More Trig Functions

Pre/Calculus 11, Veritas Prep.

We know how to find trig functions of certain, special angles. Using our unit circle definition of the trig functions, as well as our knowledge of a couple special right triangles, we can find the sine/cosine/tangent of angles like π, 2π, 3π, $-\pi$, -2π—all the multiples of π—as well as all the multiples of $\pi/2$, $\pi/3$, etc.—really, all the multiples of

$$\pi, \quad \frac{\pi}{2}, \quad \frac{\pi}{3}, \quad \frac{\pi}{4}, \quad \frac{\pi}{6}$$

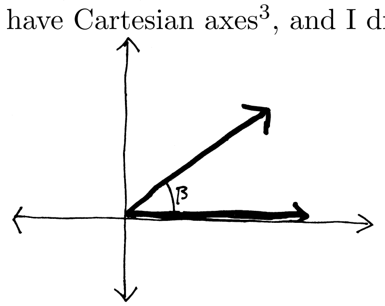
But it would be nice if we could evaluate the trig functions of even more angles!!! So... well, I don't know how we would do this in general. But let me suggest one way that maybe we could go about adding one more angle to our inventory. Here's something I notice when I look at that list. What happens if I subtract $\pi/3$ and $\pi/4$?

$$\frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi - 3\pi}{12} = \frac{\pi}{12}$$

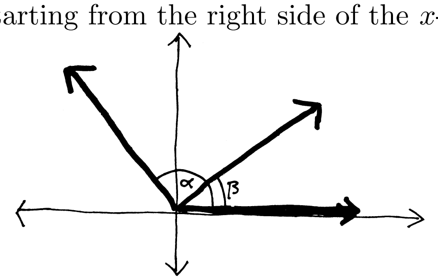
I get a new angle—$\pi/12$—an angle that isn't already on the list[1]. So maybe... I mean, we don't know what, like, $\sin(\pi/3 - \pi/4)$ is. We don't know any way to simplify that. It's certainly not just $\sin(\pi/3) - \sin(\pi/4)$. We can't just break trig functions up like that. But maybe... maybe if we could come up with some sort of formula for, like, the sine and cosine of (something – something else)... maybe then I could be able to work out $\sin(\pi/3 - \pi/4)$, and by extension, $\sin(\pi/12)$. I guess what I mean, more formally, is: can we come up for a formula for $\sin(\alpha - \beta)$ (and $\cos(\alpha - \beta)$) only using $\sin(\alpha)$, $\sin(\beta)$, $\cos(\alpha)$, $\cos(\beta)$, and ordinary arithmetic? Because if we could do that, we could figure out the sine and cosine of $\pi/12$. Which would be one little step towards our ultimate goal of being able to find the sine and cosine of any angle!!!

Let's do it. Let's try to find a formula for $\cos(\alpha - \beta)$. The basic idea of this derivation[2] is that we'll draw the angle $\alpha - \beta$ in two different ways. Then we'll use the Pythagorean theorem/distance formula to translate these two geometric pictures into algebraic sentences (i.e., equations). Then, because our two pictures are pictures of the same angle, we'll be able to set these two equations equal to each other, do some algebra, and ultimately solve for $\cos(\alpha - \beta)$.

Here's how we'll start. Imagine I have Cartesian axes[3], and I draw the angle β on them:



And then, on the same axes, also starting from the right side of the x-axis, I draw the angle α:

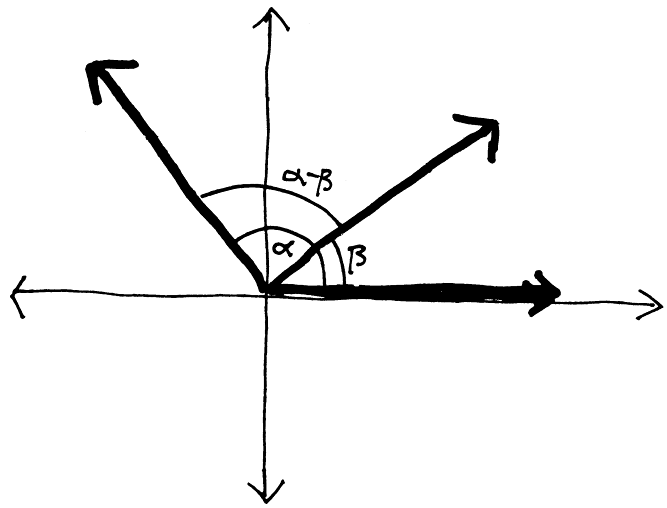


Obviously, α and β could be anything; they're not specific angles. The cool observation here is that this picture contains the angle $\alpha - \beta$! See it? It's just the angle in between α and β:

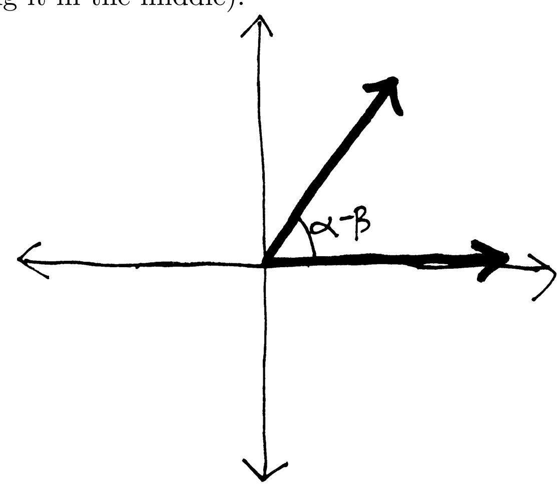
[1] In number-theory terms, the reason this happens is because 4 and 3 are "relatively prime"—they share no common factors. And so when I go to find a common denominator, I have to multiply them together.

[2] Which is just a polysyllabic word for "proof," or, more accurately, for a certain type of proof.

[3] Just a fancy name for "a graph with x and y axes," after Rene Descartes.

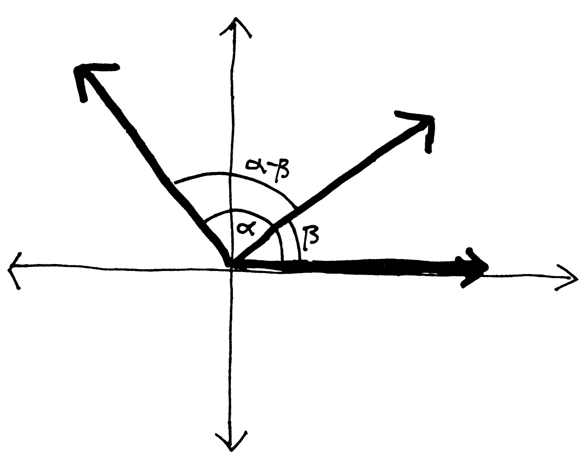


Meanwhile, I can also draw the angle $\alpha - \beta$ in the "normal" way, meaning that I can draw it starting from the x-axis (rather than drawing it in the middle):

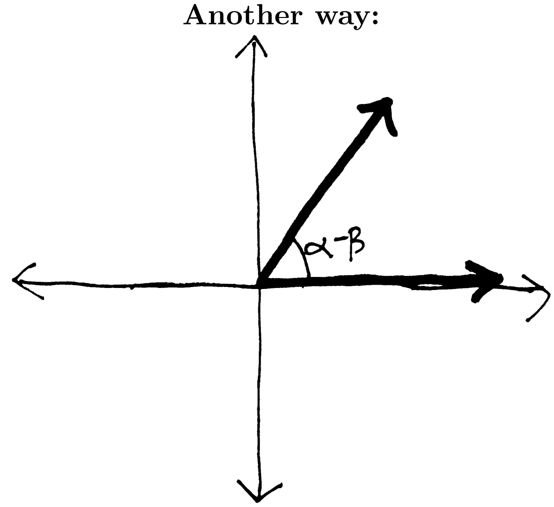


So then I have two different ways of drawing the same angle!

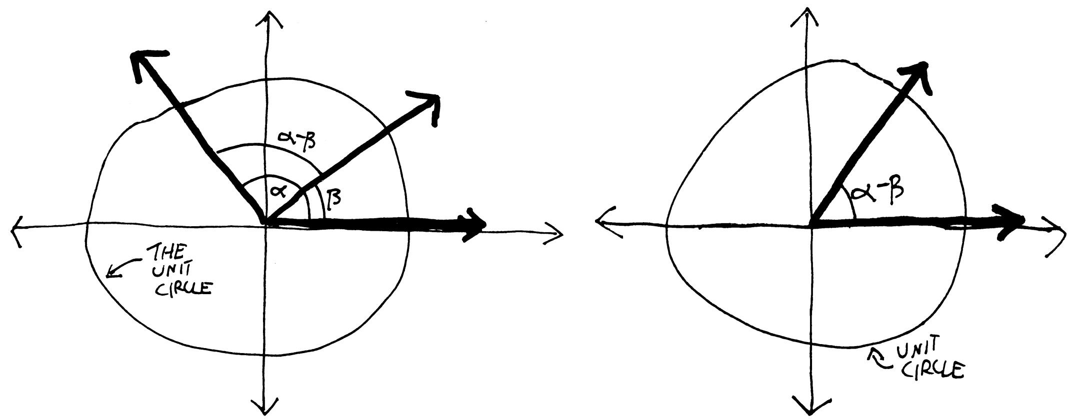
One way:



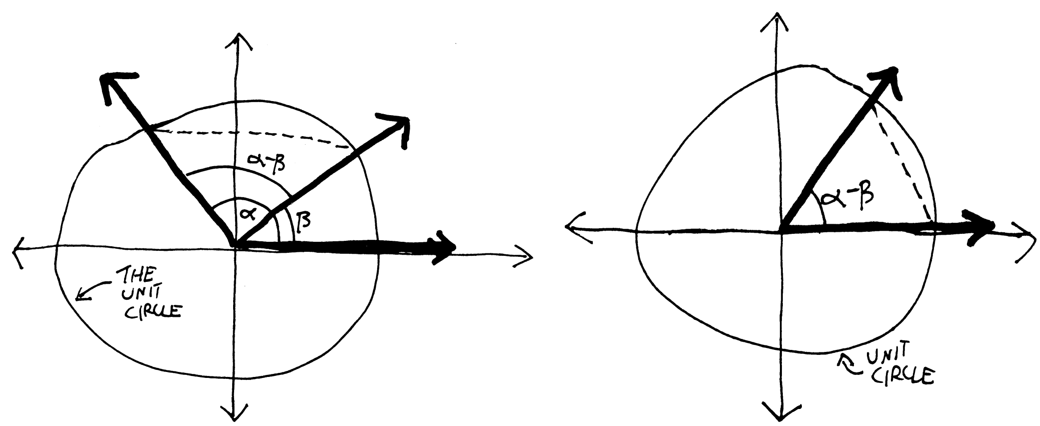
Another way:



Here's the interesting thing. What if I draw the unit circle on these graphs, like so:

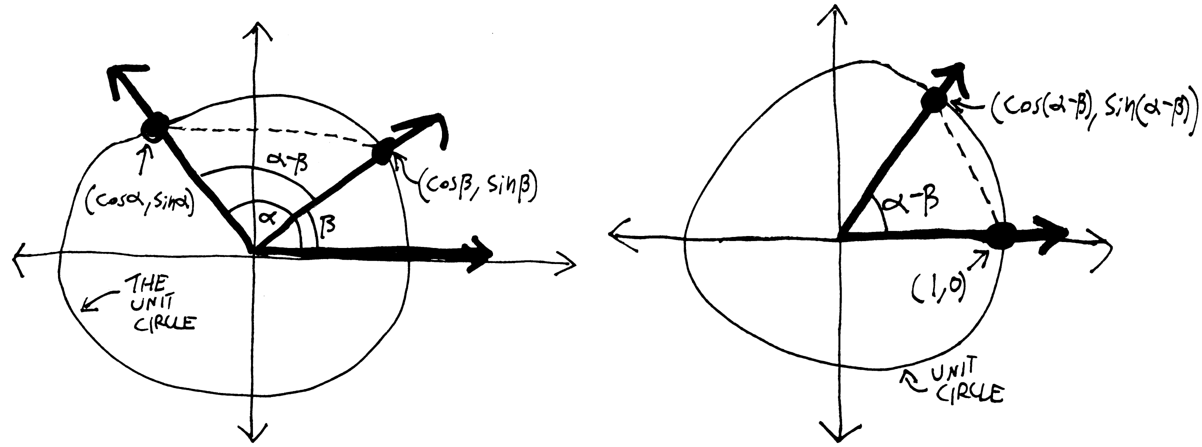


Then I can notice that the following line (opposite the angle $\alpha - \beta$)(the dotted one) is the same line in both drawings:



The line is in a slightly different position, yes—but it must be the same length. The only difference in the two pictures is that we've rotated the angle $\alpha - \beta$. The dotted line is just the other side of this triangle formed by these two radii of length 1 and the angle $\alpha - \beta$ (side-angle-side, anyone?). So whatever the length of the line is, it must be the same for both drawings.

If only we had a way to measure the length! But we do. Because this is a unit-circle setup, we know the coordinates of the two ends of the line:



So we can use the distance formula to find the lengths of the line! The equations will look different, but we know that the triangles are the same, so they'll have to work out to be the same thing (and then we'll be able to set the two different equations for the length of the dotted line equal to each other, etc., and eventually solve for just $\cos(\alpha - \beta)$).

The distance formula, keep in mind, is just a modified version of the Pythagorean theorem. If we have two points (x_1, y_1) and (x_2, y_2), then the distance between them (or the length of a straight line between them) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

So let's apply this to the diagram on the left. If I use the distance formula to find the length of the dotted line, I get:

$$\text{length} = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\beta - \sin\alpha)^2}$$

Then if I multiply out the first square, this becomes:

$$\text{length} = \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + (\sin\beta - \sin\alpha)^2}$$

And if I multiply out the next square, I get:

$$\text{length} = \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\beta - 2\sin\alpha\sin\beta + \sin^2\alpha}$$

Messy, messy, messy. And long. But we can make it shorter! Remember that thing we proved the other day? The Pythagorean Identity?[4] Remember the Pythagorean identity? The one about $\sin^2\theta + \cos^2\theta = 1$? We can use that here! Allow me to rearrange this equation slightly:

$$\text{length} = \sqrt{(\cos^2\alpha + \sin^2\alpha) - 2\cos\alpha\cos\beta + (\cos^2\beta + \sin^2\beta) - 2\sin\alpha\sin\beta}$$

See? Parentheses? We have $\cos^2\alpha + \sin^2\alpha$, which must be equal to just 1, and we also have $\cos^2\beta + \sin^2\beta$, which must also be equal to 1. So we have:

$$\text{length} = \sqrt{1 - 2\cos\alpha\cos\beta + 1 - 2\sin\alpha\sin\beta}$$

or if I combine the 1's:

$$\text{length} = \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$$

So this is ONE way of writing the length of that dotted line. Phew. ALTERNATIVELY, we could find the length using the picture on the RIGHT. If we do that, and if we apply the distance formula, we get:

$$\text{length} = \sqrt{(\cos(\alpha - \beta) - 1)^2 + (\sin(\alpha - \beta) - 0)^2}$$

You may want to glance back at the picture to convince yourself that this is true. Then, if I multiply out those squares, I get:

$$\text{length} = \sqrt{\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)}$$

Kinda messy. But AGAIN, I can apply the Pythagorean Identity!!! (I TOLD you it would be useful!)[5] We have $\cos^2(\alpha - \beta)$ and also $\sin^2(\alpha - \beta)$, so when added together these must be 1. So I must have:

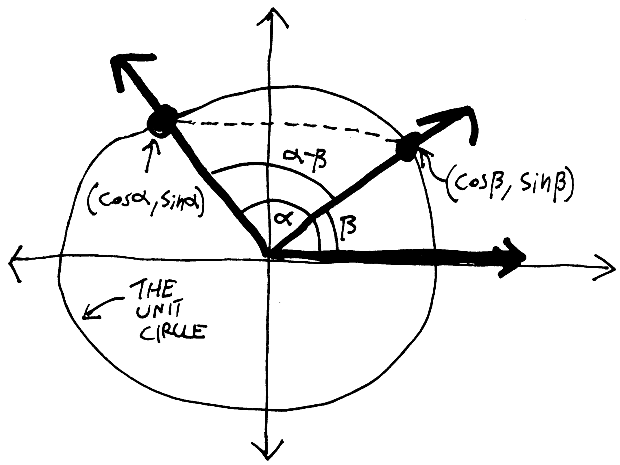
$$\text{length} = \sqrt{-2\cos(\alpha - \beta) + 1 + 1}$$

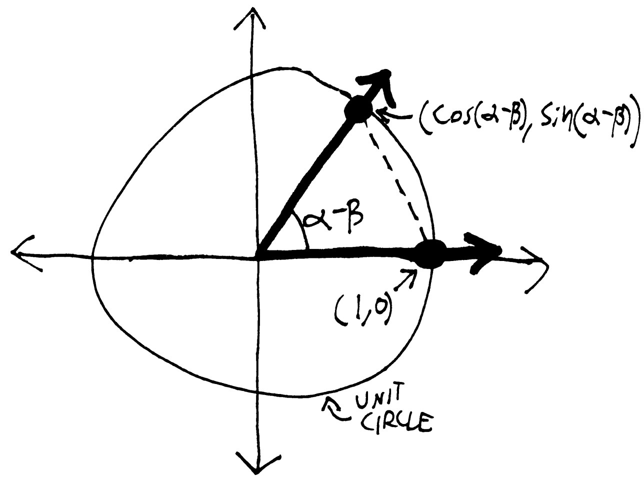
$$\text{length} = \sqrt{2 - 2\cos(\alpha - \beta)}$$

So we have two different ways to write the length of the same line. Let's summarize:

[4] The thing I love about this proof, by the way, is how long and intricate it is and how it pulls together so many different concepts.

[5] Sorry for all the capitals. It's 12:15 AM, and I a) have to finish this before I go to bed, b) have to wake up at 5:30, and c) have had lots of caffeine due to a).





Length of dotted line: $\sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$

Length of dotted line: $\sqrt{2 - 2\cos(\alpha - \beta)}$

Obviously, these two lines are the same lines. So their lengths must be equal. So I can set these two equations equal to each other, and (hopefully!) solve for $\cos(\alpha - \beta)$!!!

$$\begin{aligned}
\sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} &= \sqrt{2 - 2\cos(\alpha - \beta)} && \\
2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= 2 - 2\cos(\alpha - \beta) && \text{(squaring both sides)}\\
-2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta &= -2\cos(\alpha - \beta) && \text{(subtracting 2)}\\
\cos\alpha\cos\beta\sin\alpha\sin\beta &= \cos(\alpha - \beta) && \text{(dividing by } -2)\\
\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta &&
\end{aligned}$$

There we go! We've done it! We've found an equation for $\cos(\alpha - \beta)$! This means that we can now find $\cos(\pi/12)$!!! We can use the equation that we just derived:

$$\begin{aligned}
\cos(\pi/12) &= \cos(\pi/3 - \pi/4) && \text{(fractions)}\\
&= \cos(\pi/3)\cos(\pi/4) + \sin(\pi/3)\sin(\pi/4) && \text{(by our equation!!!)}\\
&= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} \quad + \quad \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} && \text{(trig)}\\
&= \frac{1+\sqrt{3}}{2\sqrt{2}} && \text{(fractions)}
\end{aligned}$$

OMG!!!!1!!!!1111 We've found the cosine! $\cos\left(\frac{\pi}{12}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}$!

But there are still many questions unanswered. We know the cosine of $\pi/12$—but what's the sine? We know $\cos(\alpha - \beta)$—but what about $\cos(\alpha + \beta)$? Or $\sin(\alpha + \beta)$? Or $\sin(\alpha - \beta)$? We could make a whole list of related formulae:

Sum and Difference Identities:
$\sin(\alpha + \beta) =$
$\sin(\alpha - \beta) =$
$\cos(\alpha + \beta) =$
$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

We only know one of these formulae. Let's see if we can find the other three (with the goal, ultimately, to be able to fine $\sin(\alpha-\beta)$, and thus $\sin(\pi/12)$).

Now, we could follow the same procedure that we did for these other three formulae. It would work. But it was a rather tedious, rather long process. It would be nice if there was an easier way. There is! Now that we've done the hard work to get one of the formulae, the others come much more easily. (In class I described this with an extended metaphor about burglary that I don't really have time to scribble down.)

For instance, if we want to find $\cos(\alpha+\beta)$, we can use the formula we already have—we can just "subtract" "negative beta", and then use two of our symmetry identities to simplify it:

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos(\alpha-(-\beta)) && \text{(just algebra!!!)}\\
&= \cos(\alpha)\cos(-\beta)+\sin(\alpha)\sin(-\beta) && \text{(we can apply our equation)}\\
&= \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(-\beta) && \text{(we know } \cos(-\theta)=\cos(\theta))\\
&= \cos(\alpha)\cos(\beta)+\sin(\alpha)(-\sin(\beta)) && \text{(and } \sin(-\theta)=-\sin(\theta))\\
&= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) && \text{(distributing negative)}
\end{aligned}$$

What about the sine? Here's the tricky thing: we know that sine and cosine *are the same function...* just modulo a horizontal shift[6]. More formally, we know that $\sin(\theta)=\cos(\theta-\pi/2)$. So if we think of a sine as just being a shifted cosine, we can have:

$$\sin(\alpha+\beta)=\cos(\,(\alpha+\beta)\,-\pi/2)$$

And I can move the parentheses around and write this as:

$$\sin(\alpha+\beta)=\cos(\alpha+(\beta-\pi/2)\,)$$

But then we can just apply our formula for the cosine of a sum! We have two parts—α, and $\beta-\pi/2$. So if we plug those into the formula, we get:

$$\sin(\alpha+\beta)=\cos(\alpha)\cos(\beta-\pi/2)-\sin(\alpha)\sin(\beta-\pi/2)$$

Which is kind of ugly. We could simplify the $\cos(\beta-\pi/2)$ by applying our formula for $\cos(\alpha-\beta)$ again, but we don't need to. This is just that same horizontal shift identity!!! We know that $\sin(\theta)=\cos(\theta-\pi/2)$. We've already used it, even. We can apply it AGAIN. So then we have:

$$\sin(\alpha+\beta)=\cos(\alpha)\sin(\beta)-\sin(\alpha)\sin(\beta-\pi/2)$$

What about this $\sin(\beta-\pi/2)$? Can we simplify that? You might not know it offhand, but we can come up with a similar horizontal shift identity. You can convince yourself that $\sin(\theta-\pi/2)=-\cos(\theta)$. So if we apply that, we have:

$$\sin(\alpha+\beta)=\cos(\alpha)\sin(\beta)-\sin(\alpha)(-\cos(\beta)\,)$$

which, if we distribute the negative, becomes just:

$$\sin(\alpha+\beta)=\cos(\alpha)\sin(\beta)+\sin(\alpha)\cos(\beta)$$

Okay. But we still want to be able to find $\sin(\pi/12)$. And in order to do that, we'll need to find $\sin(\pi/3-\pi/4)$, and in order to do that, we'll need to have a formula for $\sin(\alpha-\beta)$. Luckily, we can just use the same method (or a similar method) we used a little bit ago:

[6]The word "modulo" in this context means "they are the same except for a horizontal shift." Another similar usage might be, "my blue 1999 minivan is the same as your blue 2001 minivan, modulo the model year." I have been part of a small cult movement that has been trying, for the past few years, to introduce this word (originally from math) into the popular lexicon. Here's another example, from an email my dad sent a few days ago: "Actually, life in Switzerland is not so different from life at home, modulo a few obvious things like living without a car, struggling to communicate with people on the street, and being dirt poor."

$$
\begin{aligned}
\sin(\alpha-\beta) &= \sin(\alpha+(-\beta)) && \text{(just algebra!!!)}\\
&= \cos(\alpha)\sin(-\beta)+\sin(\alpha)\cos(-\beta) && \text{(by the formula we just came up with)}\\
&= \cos(\alpha)\sin(-\beta)+\sin(\alpha)\cos(\beta) && \text{(we know } \cos(-\theta)=\cos(\theta)\text{)}\\
&= \cos(\alpha)(-\sin(\beta))+\sin(\alpha)\cos(\beta) && \text{(and } \sin(-\theta)=-\sin(\theta)\text{)}\\
&= -\cos(\alpha)\sin(\beta)+\sin(\alpha)\cos(\beta) && \text{(distributing negative)}\\
&= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta) && \text{(rearranging)}
\end{aligned}
$$

So there we have it: all of our sum and difference identities!!! Let's summarize:

Sum and Difference Identities:
$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$
$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$
$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$
$\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$

Formulae that follow like a tedious argument... we still have this overwhelming question: what's the sine of $\pi/12$? Now we can figure it out!!! Using the same method that we used to find $\cos(\pi/12)$, we can split $\pi/12$ up into $\pi/3-\pi/4$, and then apply the identity we just derived:

$$
\begin{aligned}
\sin(\pi/12) &= \sin(\pi/3-\pi/4) && \text{(fractions)}\\
&= \sin(\pi/3)\cos(\pi/4)-\cos(\pi/3)\sin(\pi/4) && \text{(by our equation!!!)}\\
&= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} \quad - \quad \frac{1}{2}\cdot\frac{1}{\sqrt{2}} && \text{(trig)}\\
&= \frac{-1+\sqrt{3}}{2\sqrt{2}} && \text{(fractions)}
\end{aligned}
$$

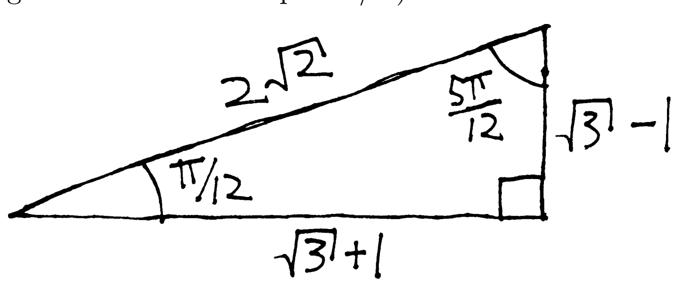
Whoopee!!!! Now we know both the sine and cosine of $\pi/12$! But wait: something's interesting here. We know

$$\cos(\pi/12)=\frac{1+\sqrt{3}}{2\sqrt{2}} \qquad \text{and} \qquad \sin(\pi/12)=\frac{-1+\sqrt{3}}{2\sqrt{2}}$$

And we also know that, at least with angles between 0 and $\pi/2$, cosine and sine are the ratios of the adjacent and opposite sides of a right triangle to its hypotenuse!

$$\cos(\pi/12)=\frac{1+\sqrt{3}}{2\sqrt{2}}=\frac{\text{adj}}{\text{hyp}} \qquad \text{and} \qquad \sin(\pi/12)=\frac{-1+\sqrt{3}}{2\sqrt{2}}=\frac{\text{opp}}{\text{hyp}}$$

So we can construct a *new special right triangle*, whose[7] hypotenuse is $2\sqrt{2}$, whose angle adjacent to $\pi/12$ is $1+\sqrt{3}$, and whose angle opposite to $\pi/12$ is $-1+\sqrt{3}$! (Note, then, that the other angle in the triangle will be $5\pi/12$, since the angles have to all add up to $\pi/2$.) THIS IS SO COOL.



[7] I know I should be writing "the hypotenuse of which is...", but I'd rather anthropomorphize my triangles.

So **this is what all of our work comes down to**. We did this huge derivation to find the sum and difference identities, and then we used the sum and difference identities to evaluate the sine and cosine of $\pi/12$, and then we used *that* to build this special right triangle. And now we can evaluate the sine/cosine/tangent of any multiple of $\pi/12$. Just using this new special right triangle. Phew.

This was fun.

Problems

Using what you know about trigonometry, the unit circle, and special right triangles—including our brand-new special right triangle!—evaluate the following trig functions without a calculator:

1. $\sin(\pi/12)$
2. $\cos(\pi/12)$
3. $\tan(\pi/12)$
4. $\sin(5\pi/12)$
5. $\cos(5\pi/12)$
6. $\tan(5\pi/12)$
7. $\sin(7\pi/12)$
8. $\cos(7\pi/12)$
9. $\tan(7\pi/12)$
10. $\sin(11\pi/12)$
11. $\cos(11\pi/12)$
12. $\tan(11\pi/12)$
13. $\sin(13\pi/12)$
14. $\cos(13\pi/12)$
15. $\tan(13\pi/12)$
16. $\sin(17\pi/12)$
17. $\cos(17\pi/12)$
18. $\tan(17\pi/12)$
19. $\sin(19\pi/12)$
20. $\cos(19\pi/12)$
21. $\tan(19\pi/12)$
22. $\sin(23\pi/12)$
23. $\cos(23\pi/12)$
24. $\tan(23\pi/12)$
25. $\sin(25\pi/12)$
26. $\cos(25\pi/12)$
27. $\tan(25\pi/12)$
28. $\sin(9,456,342\pi)$
29. $\cos(456,093,235\pi)$
30. $\tan(300,564,222\pi)$
31. $\sin(349\pi)$
32. $\cos(557,563\pi)$
33. $\tan(1137\pi)$