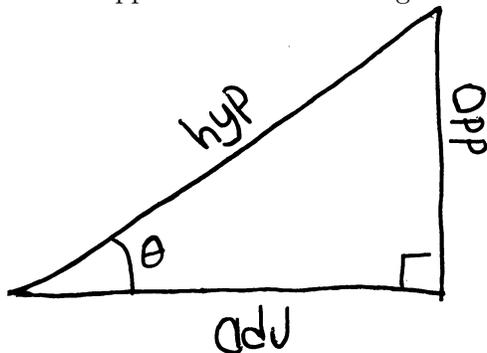


# The Unit Circle Definition of Trig Functions

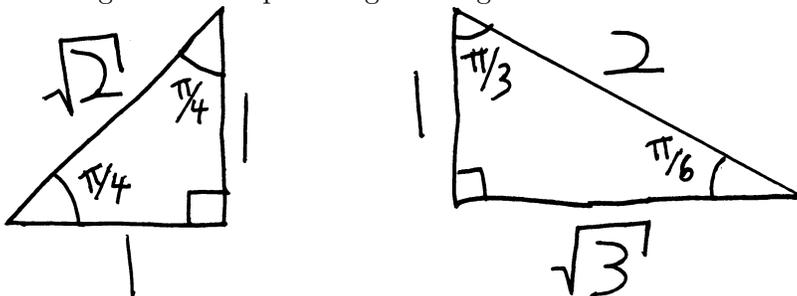
Pre/Calculus 11, Veritas Prep.

Thanks to your previous experiences in math and physics, you already have some basic knowledge of trigonometric functions—sine, cosine, tangent, and the like. You've probably defined them in terms of a right triangle: for example, if you have some right triangle, and some angle  $\theta$  in that triangle, then the sine of  $\theta$  is the ratio of the length of the opposite side to the length of the hypotenuse.



$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} \quad \cos(\theta) = \frac{\text{adj}}{\text{hyp}} \quad \tan(\theta) = \frac{\text{opp}}{\text{adj}}$$

These are useful, because they are the Rosetta Stone that allows us to go from talking about angles to talking about distances (from circular rotation to linear motion). In math and physics, you've probably used calculators to find sines and cosines most of the time, but you've probably seen that there are some special angles, like  $\pi/6 = 30^\circ$ , that you can find the sine/cosine/tangent of exactly, and not as a decimal approximation. You can do this using these two special right triangles:



For example, you can exactly evaluate  $\sin(\pi/3)$  by looking at the special right triangle with a  $\pi/3$  angle, seeing that the side opposite to that angle has length  $\sqrt{3}$ , and the hypotenuse of that triangle has length 2. So then

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

And it is much nicer to write  $\sqrt{3}/2$  than it is to write 0.866025404, which is what your calculator would tell you.

Now, using this method, we still might not be able to calculate every trig function nicely. For example, what's  $\sin(7\pi/34)$ ? We don't know, because we don't know any special right triangles with a  $7\pi/34$  angle in them. Like with logarithms, we are restricted to only evaluating some trig functions exactly and beautifully. The others we can use a calculator for. Your calculator, of course, uses formulas that fill in the gaps to estimate trig functions/logs as decimal approximations, but those formulas aren't particularly interesting.

Our real issue now is this one: our definition of these trig functions is based on triangles. Specifically, on the angles in, and the ratio of the sides of, a right triangle. But the angles in a right triangle have to be between  $0$  and  $90^\circ$ . What if we want to find the sine of some angle greater than  $90^\circ$ ? We can't make a right triangle that has an angle greater than  $90^\circ$ , so we can't find the sin/cosine/tangent of any angle greater than  $90^\circ$ . What if we want to find the sine/cosine/tangent of a negative angle? We certainly can't

make a right triangle with a negative angle. (Do negative angles even exist? is it even reasonable to talk about such things?) This is the downside to our current definition of trig functions. It restricts us to only finding trig functions of angles between 0 and 90 degrees.

What if we wanted to extend our definition so that we could find the sine/cosine/tangent of angles less than  $0^\circ$ , and greater than  $90^\circ$ ?

special right triangles

examples

e.g.

we still might not be able to calculate every single trig function as a decimal—what's  $\sin(7\pi/34)$ ?—but we can compute a few. just like with logs .

anyway, it's easy to come up with formulas that can fill in the gaps and approximate to find things as decimals like  $\sin(7\pi/34)$ , but that's not particularly interesting (not right now, at least).

the real question is this: our definition of these trig functions is based on triangles. on the angles of a right triangle. but the angles in a right triangle have to be between 0 and  $90^\circ$ .

but want to extend. what if we want to take the sine of  $-\pi/3$ ? or the cosine of  $5\pi/4$ ? we can't draw a triangle that has a negative angle (that'd be like drawing a negative distance!) we can't draw a right triangle that has an angle of more than 90 degrees in it. *can* we even take the sine/cosine/tangent of these angles?

extend definition of exponents

extend definition of trig functions!

some sort of coordinate system so that having a "negative angle" makes sense (in the same way that we can have a "negative distance"—just go in opposite direction)

so put a triangle on the coordinate system! if we do this, then one thing we notice is:

i can draw a whole bunch of right triangles, but i only care about similar ones, so let's keep hypotenuse the same

and why not choose 1?

and some way of measuring a triangle

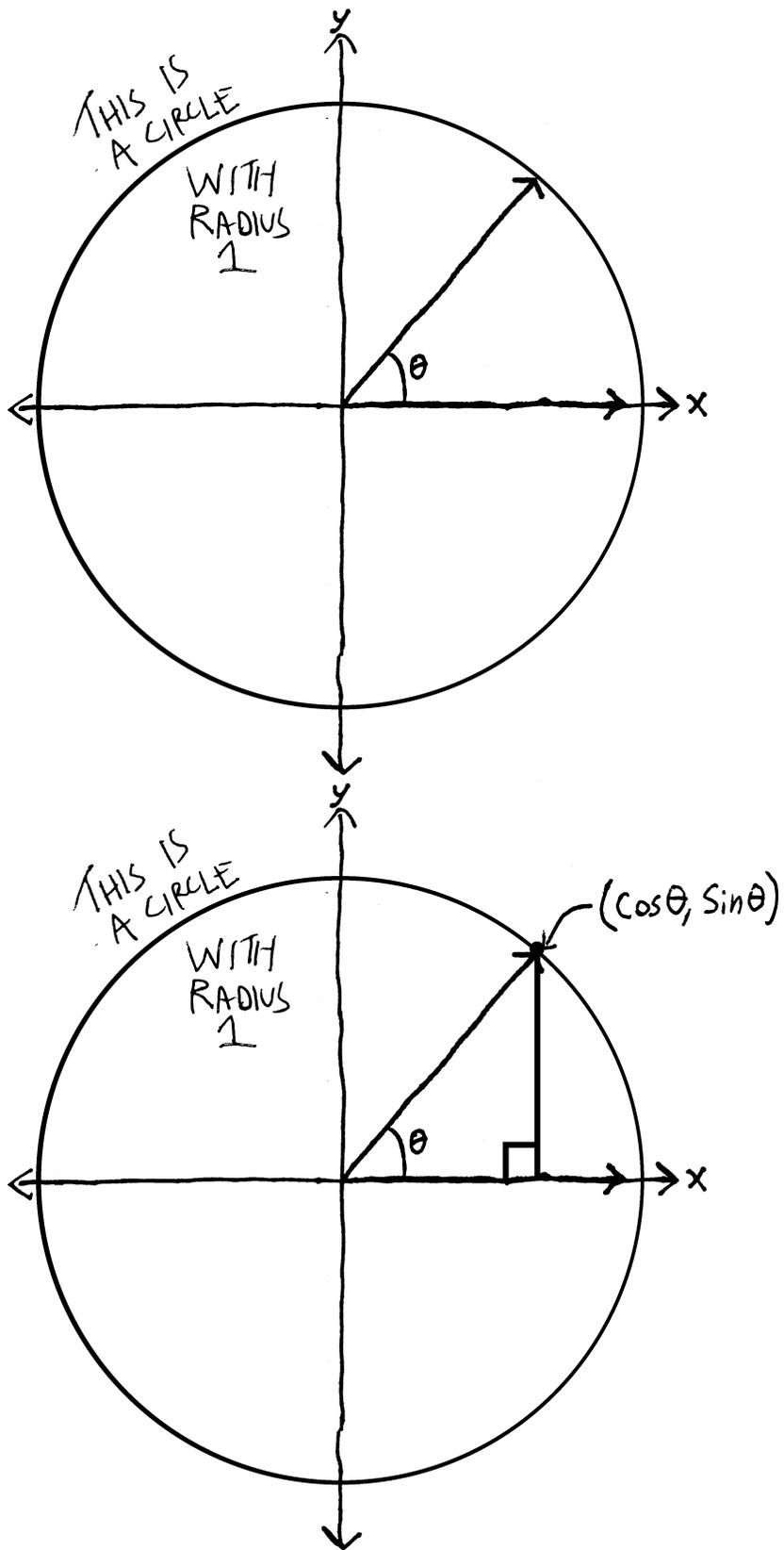
thus unit circle

examples with working it out.

EG find  $\sin(\pi/3)$

EG find  $\cos(5\pi/6)$

EG find  $\sin(-\pi/4)$



clearly, this matches up with our old definition of trig functions. in our old definition, sine was opp/hyp. in this case,  $\sin(\theta) = \text{opp}/1 = \text{opp}$ , which is the same as the  $y$ -coordinate of that point. likewise with  $\cos(\theta)$ .

## Problems

Using what you know about trigonometry, special right triangles, and the unit circle, evaluate the following trig functions without a calculator:

- |                   |                     |                    |                    |
|-------------------|---------------------|--------------------|--------------------|
| 1. $\sin(0)$      | 22. $\sin(-3\pi)$   | 43. $\sin(\pi/4)$  | 64. $\sin(4\pi/3)$ |
| 2. $\cos(0)$      | 23. $\cos(-3\pi)$   | 44. $\cos(\pi/4)$  | 65. $\cos(4\pi/3)$ |
| 3. $\tan(0)$      | 24. $\tan(-3\pi)$   | 45. $\tan(\pi/4)$  | 66. $\tan(4\pi/3)$ |
| 4. $\sin(\pi)$    | 25. $\sin(-4\pi)$   | 46. $\sin(3\pi/4)$ | 67. $\sin(5\pi/3)$ |
| 5. $\cos(\pi)$    | 26. $\cos(-4\pi)$   | 47. $\cos(3\pi/4)$ | 68. $\cos(5\pi/3)$ |
| 6. $\tan(\pi)$    | 27. $\tan(-4\pi)$   | 48. $\tan(3\pi/4)$ | 69. $\tan(5\pi/3)$ |
| 7. $\sin(2\pi)$   | 28. $\sin(\pi/6)$   | 49. $\sin(7\pi/4)$ | 70. $\sin(7\pi/3)$ |
| 8. $\cos(2\pi)$   | 29. $\cos(\pi/6)$   | 50. $\sin(5\pi/4)$ | 71. $\cos(7\pi/3)$ |
| 9. $\tan(2\pi)$   | 30. $\tan(\pi/6)$   | 51. $\cos(5\pi/4)$ | 72. $\tan(7\pi/3)$ |
| 10. $\sin(3\pi)$  | 31. $\sin(5\pi/6)$  | 52. $\tan(5\pi/4)$ | 73. $\sin(\pi/2)$  |
| 11. $\cos(3\pi)$  | 32. $\cos(5\pi/6)$  | 53. $\cos(7\pi/4)$ | 74. $\cos(\pi/2)$  |
| 12. $\tan(3\pi)$  | 33. $\tan(5\pi/6)$  | 54. $\tan(7\pi/4)$ | 75. $\tan(\pi/2)$  |
| 13. $\sin(4\pi)$  | 34. $\sin(7\pi/6)$  | 55. $\sin(9\pi/4)$ | 76. $\sin(3\pi/2)$ |
| 14. $\cos(4\pi)$  | 35. $\cos(7\pi/6)$  | 56. $\cos(9\pi/4)$ | 77. $\cos(3\pi/2)$ |
| 15. $\tan(4\pi)$  | 36. $\tan(7\pi/6)$  | 57. $\tan(9\pi/4)$ | 78. $\tan(3\pi/2)$ |
| 16. $\sin(-\pi)$  | 37. $\sin(11\pi/6)$ | 58. $\sin(\pi/3)$  | 79. $\sin(5\pi/2)$ |
| 17. $\cos(-\pi)$  | 38. $\cos(11\pi/6)$ | 59. $\cos(\pi/3)$  | 80. $\cos(5\pi/2)$ |
| 18. $\tan(-\pi)$  | 39. $\tan(11\pi/6)$ | 60. $\tan(\pi/3)$  | 81. $\tan(5\pi/2)$ |
| 19. $\sin(-2\pi)$ | 40. $\sin(13\pi/6)$ | 61. $\sin(2\pi/3)$ |                    |
| 20. $\cos(-2\pi)$ | 41. $\cos(13\pi/6)$ | 62. $\cos(2\pi/3)$ |                    |
| 21. $\tan(-2\pi)$ | 42. $\tan(13\pi/6)$ | 63. $\tan(2\pi/3)$ |                    |