PS#15 select solutions

Math 3

5-49: Some proofs with the modulus (intentionally out of order)

Let's talk about the radius/modulus/magnitude of a complex number a little bit! We didn't introduce any special notation for it, but the notation that often gets used is the same as absolute value notation:

$$|z| \equiv \text{ the distance } z \text{ is away from the origin/its radius/magnitude/modulus}$$

In other words:

$$\text{if } z = a + bi, \text{ then } |z| = \sqrt{a^2 + b^2}$$
$$\text{if } z = r(\cos\theta + i\sin\theta), \text{ then } |z| = r$$

This notation should make sense, because if z is a real number, its distance from the origin is just the absolute value. We're just generalizing it!

Here are some fun properties to prove! (Remember, to prove these equalities, start with one side of the equation, and simplify forwards until you get to the other side. It might be easier to start from the right side, or it might be easier to start from the left side!)

I'll prove these using the rectangular $a + bi$ form of a complex number, but you could prove them in a different form, too—$\cos\theta + i\sin\theta$, $e^{i\theta}$, etc.. Lots of options! (Is it easier to prove in one form than another?)

a. *Show that for all complex numbers z,* $|z| = |\overline{z}|$

Suppose $z = a + bi$. Then we have:

$$\begin{aligned} |z| &= \sqrt{a^2 + b^2} \\ &= \sqrt{a^2 + (-b)^2} \\ &= |a - bi| \\ &= |\overline{z}| \end{aligned}$$

b. *Show that for all complex numbers z,* $z\overline{z} = |z|^2$

Suppose $z = a + bi$. Then we have:

$$\begin{aligned} z\overline{z} &= (a + bi)(a - bi) \\ &= a^2 + abi - abi - b^2 i^2 \\ &= a^2 + b^2 \\ &= \left(\sqrt{a^2 + b^2}\right)^2 \\ &= |z|^2 \end{aligned}$$

c. *Show that for all complex numbers z,* $|z \cdot w| = |z| \cdot |w|$

Suppose $z = a + bi$ and $w = c + di$. Then we have:

$$\begin{aligned}
|z \cdot w| &= |(a+bi)(c+di)| \\
&= \left|ac + adi + bci + bdi^2\right| \\
&= |ac + adi + bci - bd| \\
&= |(ac - bd) + (ad + bc)i| \\
&= \sqrt{(ac-bd)^2 + (ad+bc)^2} \\
\text{oh man, this isn't going to be fun} & \\
&= \sqrt{(a^2c^2 - 2abcd + b^2d^2) + (a^2d^2 + 2abcd + b^2d^2)} \\
&= \sqrt{a^2c^2 + b^2d^2a^2d^2 + b^2d^2} \\
\text{good gracious, maybe I'll try to back-solve on} & \\
\text{some scratch paper to figure out where to go} & \\
&= \sqrt{(a^2+b^2)\left(c^2+d^2\right)} \\
&= \sqrt{a^2+b^2} \cdot \sqrt{c^2+d^2} \\
&= |z| \cdot |w|
\end{aligned}$$

Egads, that was some unpleasant algebra. Let's see if it's easier in polar. Suppose $z = r_z e^{i\theta_z}$ and $w = r_w e^{i\theta_w}$. Then:

$$\begin{aligned}
|z \cdot w| &= \left|r_z e^{i\theta_z} r_w e^{i\theta_w}\right| \\
&= \left|r_z r_w e^{i(\theta_z + \theta_w)}\right| \\
&= r_z \cdot r_w \\
&= |z| \cdot |w|
\end{aligned}$$

Yeah, just a little easier.

d. *True or false?* $|z| \geq |Re(z)|$. *Why?*

True. z's distance to the origin will always be at least as big as its distance along the real axis. It'll be equal to its distance along the real axis if z itself is real. Otherwise, it'll be bigger.

e. *True or false?* $|z| \geq |Im(z)|$. *Why?*

True. z's distance to the origin will always be at least as big as its distance along the imaginary axis. It'll be equal to its distance along the imaginary axis if z itself is imaginary. Otherwise, it'll be bigger.

5-48: Adventures With George Jemmott in The $\sqrt{-1}$·Lab.

OH NO!!!! You've been blinded by the beauty of complex numbers, and now you can no longer discern their angle. Two complex numbers with the same magnitude (but different angles) look exactly the same to you! $+5$ *and* -5*—they're exactly the same!* $12i$ *and* $-12i$*—the same!* $2\angle\frac{\pi}{4}$ *and* $2\angle\frac{\pi}{6}$*—the same!*

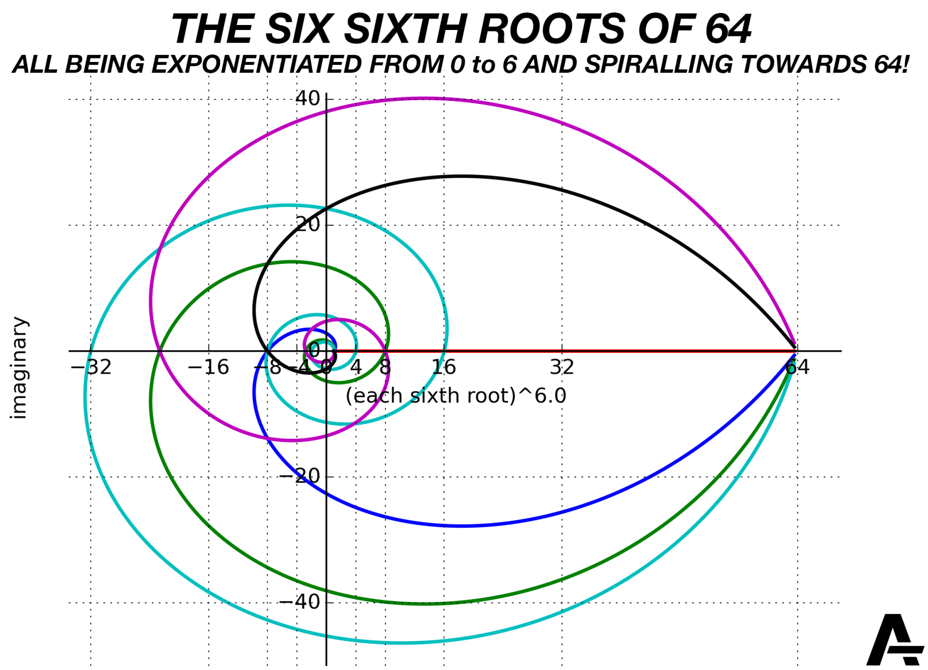
Fumbling around in the metaphorical darkness, you come across your box full of the sixth roots of 64. The box has a whole bunch of copies of the sixth roots (multiple copies of each). Can you tell them apart from each other? Why or why not?

You start randomly pulling out sixth roots and multiplying them together. If you multiply six of them together, what do you see? You don't know which *sixth roots you've multiplied together; just that you multiplied six of them together. Maybe it was six copies of the same sixth root; maybe it was three copies of one sixth root and three copies of a different sixth root; maybe it was one copy of each sixth root; maybe it was some other combination. What happens??*

(Inspired by a conversation in the i-Lab with George Jemmott. Or, should I say, the $\sqrt{-1}$·Lab.)

You can't tell the six distinct sixth roots of 64 apart from each other. They all have a radius of 2, so they all look identical to you. When you multiply them together, six of them, in whatever combination, they'll always multiply up to something that has a radius of 64. It won't be the real number 64, necessarily, but it will be *somewhere* on the circle with radius 64.

George's idea was an elaboration on my animation of the six sixth roots of 64 all spiraling out to 64:



This animation shows each of the individual sixth roots, with their exponents gradually being raised to 6, e.g.:

$$(2\angle 120°)^{\text{going from 0 to 6}} \rightarrow \text{ eventually equals 64}$$

George's idea was, what if you multiply all the roots together, with various exponents, so that the sum of their exponents goes from 0 to 6, but maybe in weird permutations or combinations, for example:

$$(\text{one of the roots})^3 \cdot (\text{another root})^2 \cdot (\text{yet another root})^1 = \text{something with a radius of 64}$$

No matter what permutation of exponents, you'd end up somewhere on the circle with radius 64 (probably not at $z = 64$), with a different and weird spiral taking you there. And it'd be cool to visualize that!

Unused Quiz Question: Cole's Conjecture:

Cole was very excited to find that the (fraternal!) third cube root of i is just $-i$. He hypothesized that because of the cyclic nature of the powers of i, any multiple-of-three'th root of i will also *have $-i$ as one of the possibilities. Was he correct? Prove or disprove. Clearly there are* some *multiple-of-three'th roots of i that have $-i$ as a possibility (the cube root). Do all of them? Is there a similar true conjecture involving which roots of i will have $-i$ as a solution?*

This is a good observation, but it's not precisely correct. For example, it's not true that one of the 6th roots of i is $-i$, because:

$$(-i)^6 = (-1)^6 i^6 = i^6 = -1 \neq i$$

But is there some relationship? Which roots of i have $-i$ as one of their possibilities? This is the same as asking:

$$(-i)^n = i$$

There are a couple of ways we could work this out. One is to just brute force it, and start writing out all the powers of $-i$:

$$(-i)^n = \underset{n=0}{+1}, \underset{n=1}{-i}, -1, +i, +1, -i, -1, +i, +1, -i, -1, +i, \cdots$$

If you think about this geometrically, we're adding 270° at each iteration—not 90°, like when we multiply by $+i$. So, when do $+i$'s show up? They show up when:

$$(-i)^n = +1, -i, -1, \underset{n=3}{+i}, +1, -i, -1, \underset{n=7}{+i}, +1, -i, -1, \underset{n=11}{+i}, \cdots$$

$$(-i)^n = i \text{ when } n = 3, 7, 11, \cdots$$

So it repeats every 4 iterations, and in particular, it repeats when the remainder divided by 4 is 3. So, in conclusion, any $4k+3$'th root of i, where k is some integer (or, equivalently, any $4k-1$'th root of i) will have $-i$ as one of the solutions.

So, that's one way to solve it. Here's another way. If you want to think about it a bit more algebraically, you could set up an equation. We want to solve this equation for k:

$$(1\angle 90°)^k = 1\angle 270°$$

So this becomes:

$$90k = 270$$

But actually, we can have extra multiples of 360°, so really this should be:

$$90k = 270 + 360n \quad \text{for any } n \in \mathbb{Z}$$

So:

$$k = \frac{270 + 360n}{90}$$

Or just:

$$k = 3 + 4n$$

You may recall the BBC documentary we watched on Fermat's Last Theorem, and the way in which the Japanese mathematician Goro Shimura described his deceased colleague Yutaka Taniyama:

> Taniyama was not a very careful person as a mathematician. He made a lot of mistakes. But he made mistakes in a good direction. So eventually he got to the right answers. I tried to imitate him, but I found out that it is very difficult to make good mistakes.

Cole's Conjecture, like Maya's Musing below, was a good mistake. It's the kind of mistake we should all be striving to make!

Unused Quiz Question: Maya's Musing:

Maya found two of the roots, $\pm\frac{\sqrt{3}}{2} + \frac{1}{2}i$ in the straightforward ordinary way. She found the third root, $-i$, though, in a more novel method: by assuming that all three of the roots have to multiply together to create i:

$$(\textit{the first root}) \cdot (\textit{the second root}) \cdot (\textit{the unknown third root}) = i$$

And then assuming that the unknown root is another complex number in the form $a + bi$, giving this equation:

$$\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \cdot \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \cdot (a + bi) = i$$

And then solving for a and b. And indeed, she calculated that the third root should be $-i$!

Muse on this method. It worked in this case—will it always work? Clearly, if we take an n'th root of z, and multiply it by itself n times, we get z. That's what a root is. But is the product of all the distinct *n'th roots of z together always equal to z? Prove or disprove. Can you find a counterexample? If it's not always equal to z, is it sometimes equal to z? When?*

This is a good observation, but it's not precisely correct. It's *not* true that the product of all the n distinct n'th roots of some complex number is always the original complex number. For example, $+2$ and -2 are the two distinct square roots of $+4$, but:

$$(+2)(-2) = -4 \neq +4$$

However, it's kind of cool that this appears to work at least for the cube roots of i. So, was that just a totally random coincidence, or is there some deeper truth here? Is it sometimes the case that the product of all the n distinct n'th roots of some complex number is the original complex number? If so, when? What's the pattern?

If you play around with different examples, it's not too hard to make a guess at what's going on: **Maya's Musing seems to be true for odd roots** (like the cube roots of i) **and false for even roots** (like the square roots of 4).

Writing a more formal proof takes a bit of work, so had I written this up as a quiz question, I probably would have just asked you to come up with the conjecture. I think there's probably a nice proof/argument you could make involving symmetries or geometry—that's probably the cleanest, fastest, and most elegant proof. But I'm not totally sure how I'd phrase or argue it.

Here's my attempt at a proof. We only need to worry about the angles here—we know (in part from the George Jemmott-inspired question) that if we multiply together distinct roots, the radii all multiply together properly. Basically what we want to prove or calculate or figure out is:

- if we take all the n distinct nth roots of some complex number and multiply them together
- i.e., if we add up all their angles
- do we get back to where we started?

This doesn't work for the two distinct roots of 4 (angle $0°$), because the angle of $+2$ is $0°$, and the angle of -2 is $180°$. So we add them, and get $180°$, which is not the angle of 4.

So, what if we think about this for all the n distinct nth roots of $r\angle\theta$? The angles of those roots are:

$$\frac{\theta}{n},\ \frac{\theta+2\pi}{n},\ \frac{\theta+2\cdot 2\pi}{n},\ \frac{\theta+3\cdot 2\pi}{n},\ \cdots,\ \frac{\theta+(n-1)\cdot 2\pi}{n}$$

If we add them all up, we get:

$$\frac{\theta+(\theta+2\pi)+(\theta+4\pi)+(\theta+6\pi)+\cdots+(\theta+(n-1)\cdot 2\pi)}{n}$$

Or:

$$\frac{n\theta\ +\ 2\pi\cdot(1+2+3+\cdots+(n-1))}{n}$$

Can we simplify all those integers added up? Fun fact: the sum of the first n ~~integers~~ natural numbers is $\frac{n(n+1)}{2}$, so then the sum of the first $n-1$ integers is $\frac{n(n+1)}{2}-n$, so we can plug that in:

$$\frac{n\theta\ +\ 2\pi\cdot\left(\frac{n(n+1}{2}-n\right)}{n}$$

We can cancel out some n's:

$$\theta\ +\ 2\pi\cdot\left(\frac{n+1}{2}-1\right)$$

So we have:

$$\text{the angle of the product of the } n \text{ distinct } n\text{th roots of } r\angle\theta : \quad = \quad \theta + 2\pi \cdot \left(\frac{n+1}{2} - 1\right)$$

Our question is, when is that equal to θ itself? What values of n make Maya's Musing true?

$$\text{for what } n \text{ is: } \theta = \theta + 2\pi \cdot \left(\frac{n+1}{2} - 1\right)$$

This is the same as asking:

$$\text{for what } n \text{ is: } 0 = 2\pi \cdot \left(\frac{n+1}{2} - 1\right)$$

Well, 2π will be the same ("the same") as 0 whenever it's multiplied by an integer! So our question becomes:

$$\text{when is } \frac{n+1}{2} - 1 \text{ an integer?}$$

But subtracting 1 won't change whether it's an integer, so we have just:

$$\text{when is } \frac{n+1}{2} \text{ an integer?}$$

When is it? Whenever n is odd! If n is even, then this will be a fraction; if n is odd, we can divide 2 and get an integer.

So the somewhat more refined and accurate version of Maya's Musing is:

- The product of all the n distinct nth roots of $r\angle\theta$ is:
  - $r\angle\theta$, if n is odd
  - $r\angle(\theta + 180°)$, if n is even.

As I say, this proof is overly long—it seems like there really should be a short, elegant, beautiful proof, somehow involving geometry and the symmetries of the roots. Let me know if you have a better idea!